EDGE Magic Pyramidal Graphs

H.Velwet Getzimah[1], D. S. T. Ramesh [2]

[1] Department of Mathematics, Pope's College, Sawyerpuram affliated to Manonmaniam Sundaranar University, Tamil Nadu, India.

[2]Department of Mathematics Margoschis College, Nazareth affliated to Manonmaniam Sundaranar University, Tamil Nadu, India.

Abstract: - Let G = (V, E) be a graph with p vertices and q edges. The Edge Magic Pyramidal labeling of a graph G with p vertices and q edges is an assignment of integers from $\{1, 2, 3, \ldots, p_q\}$ to the vertices and edges of G where p_q is the q^th^ Pyramidal number so that at each edge the sum of that edge label and the labels of the vertices incident with that edge is a constant and the constant must be a Pyramidal number. In this paper we prove that the Complete bipartite graphs, Peterson graph and all Stars are Edge Magic Pyramidal graphs and introduce the concept of Edge magic Shift labeling, Edge Antipyramidal labeling. Also we investigate the Edge magic strength of certain graphs. By a graph we mean a finite, undirected graph without multiple edges or loops. For graph theoretic terminology, we refer to Harary [4] and Bondy and Murty [2]. For number theoretic terminology, we refer to M. Apostal [1].

Keywords: Pyramidal number, Antipyramidal, Stars, bipartitite.

I. INTRODUCTION

A labeling of a graph is an assignment of labels to the vertices or edges or to both the vertices and edges subject to certain conditions. Magic labelings have their origin from magic squares and it was first introduced by Sedlacek. In an Edge magic pyramidal labeling the weight of an edge is the sum of the edge label and the labels of the vertices incident with that edge. In an Edge magic pyramidal labeling the weight of each edge is a constant and the constant must be a pyramidal number. For a particular graph there are many edge magic constants. In this paper the range of the Edge magic constants are determined for certain graphs and their magic strengths are specified. Also Strong, Weak, Ideal edge magic graphs are identified and the concept of Edge magic Shift labeling, Edge Antipyramidal labeling is introduced.

II. Edge magic pyramidal labeling

Definition 2.1: A Triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If n^th^ Triangular number is denoted by T_n then $T_n = \frac{n(n+1)}{2}$. Triangular numbers are found in the third diagonal of Pascal's Triangle starting at row 3. They are 1, 3, 6, 10, 15, 21...

Definition 2.2: The sum of Consecutive triangular numbers is known as tetrahedral numbers. They are found in the fourth diagonal of Pascal's Triangle. These numbers are 1, 1 + 3, 1 + 3 + 6, 1 + 3 + 6 + 10...
(i.e.) 1, 4, 10, 20, 35…

Definition 2.3: The Pyramidal numbers or Square Pyramidal numbers are the sums of consecutive pairs of tetrahedral numbers. The following are some Pyramidal numbers.1. 1 + 4, 4 + 10, 10 + 20, ...
(i.e.) 1, 5, 14, 30, 55...

Remark 2.4: The Pyramidal numbers are also calculated by the following formula:
$p_n = \frac{n(n+1)(2n+1)}{6}$

Definition 2.5: The Edge Magic Pyramidal labeling of a Graph G=(V,E) is defined as a one-to-one function f (we call Edge Magic Pyramidal function) fromV(G)∪E(G) onto the integers$\{1,2,3,\ldots,p_q\}$ with the property that there is a constant μ_f such that f(u) + f(v) +f(uv) = μ_f where uv ∈ $E(G)$ and the constant μ_f must be a Pyramidal number. Here p_q denotes the q^th^ Pyramidal number. The constant μ_f is called as the Edge Magic constant of the given graph. The graph which admits such a labeling is called an Edge Magic Pyramidal graph.

Remark 2.6: For a graph G, there can be many Edge Magic Pyramidal functions and for each function f there is an Edge Magic constant.

Notation: The notation p_i is used for each Pyramidal number where i = 1,2,3...

Theorem 2.7: All Complete bipartite graphs $K_{m,n}$, are Edge Magic Pyramidal with $p_{m+2} \le \mu_f \le p_{m^2+i}$ for m= n where i takes the values 0,2,4,6… for each m ranging from 2,3,4…. and for m $\neq n$, μ_f approximately varies from $p_{m+2} \le \mu_f \le p_{mn+m\sim n}$. **Proof:** Let G be a Complete bipartite graph $K_{m,n}$. Let G = (V(G), E(G). Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2. Let $v_1,v_2, \ldots,v_m$ be the vertices of V_1 and $u_1,u_2,\ldots,u_n$ be the vertices of V_2. Let $e_1,e_2,\ldots,e_{mn}$ be the edges of $K_{m,n}$. Therefore we have V = $V_1 \cup V_2$. Let $|V_1(G)|$ = m and $|V_2(G)|$ = n. Hence $|V(G)|$ = m+n and $|E(G)|$ = mn. Define f:V(G)∪E(G).→ $\{1,2,3,\dots,p_q\}$ as follows:

$f(v_1) = \left[\frac{\mu_f}{2}\right]$

$f(v_i) = f(v_{i-1}) - (n+2)$ for $2 \le i \le m \ \forall \ v_i \in V_1$

Define $f(u_1) = \begin{cases} f(v_1) - 1 & if\ \mu_f\ is\ odd \\ f(v_1) - 2 & if\ \mu_f\ is\ even \end{cases}$

$f(u_i) = f(u_{i-1}) - 1$ for $2 \le i \le n \ \forall \ u_i \in V_2$

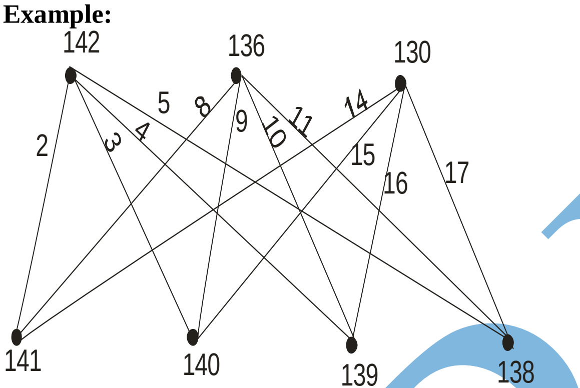
$f(e_i) = i + 1$, for $1 \le i \le mn$ and

$i \ne n+1, 2n+1, 3n+1, \dots$

$f(e_i) = f(e_{i-1}) + 3$, for $i = n+1, 2n+1, 3n+1, \dots$

By the above labeling all Complete bipartitite graphs $K_{m,n}$ are Edge Magic Pyramidal graphs.

Example:



Edge Magic Pyramidal labeling of $K_{3,4}$ (μ_f = 285)

Remark 2.8: For the above Complete bipartitite graph we have given μ_f the value p_9 = 285, We have $p_{m+2} \le \mu_f \le p_{m^2+i}$ and therefore μ_f can take all the pyramidal numbers between $p_{m+2} = p_5$ and $p_{m^2+i} = p_{13} = 819$ and hence the possible values of μ_f are 55,91,140, 204, 285, 385, 506, 650 and 819. **Theorem 2.9:** The Peterson graph is Edge Magic Pyramidal with $p_{m-3} \le \mu_f \le p_n$ where m,n are the vertices and edges in the graph.

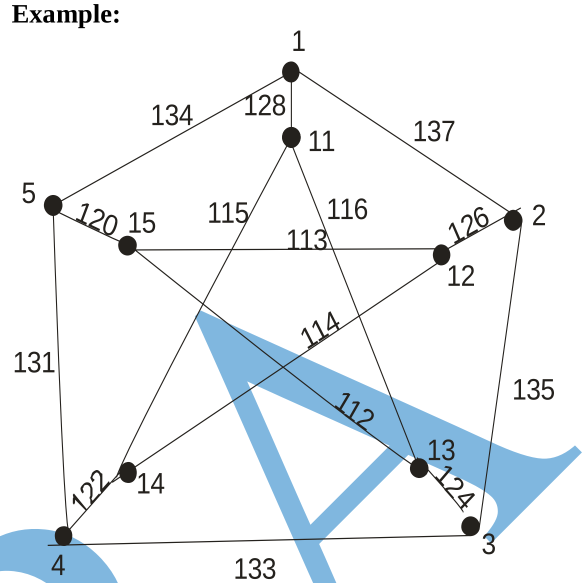
Proof: Let v_i, $1 \le i \le 10$ be the vertices in the clockwise direction. Let $e_i = v_i v_{i+1}$, $1 \le i \le 4$, $e_5 = v_5 v_1$, $e_i = v_i v_{i-5}$, $6 \le i \le 10$ be 15 be the edges in the clockwise direction. Let e_i, $11 \le i \le 15$ be the edges of the Star shaped Cycle of the Peterson graph where, $e_{11} = v_9 v_6$, in the clockwise direction. Here the number of vertices m = 10 and the number of edges n = 15.

Define $f(v_i) = \begin{cases} i & for\ 1 \le i \le 5 \\ i+5 & for\ 6 \le i \le 10 \end{cases}$

$f(e_i) = \mu_f - \sum f(v)$ for $1 \le i \le 15$ where $\sum f(v)$

denote the sum of the labels of the vertices incident with e_i. By the above labeling the Peterson graph is an Edge Magic Pyramidal graph.

Example:



Edge Magic Pyramidal labeling of Peterson graph (μ_f = 140)

Remark 2.10: For the above Peterson graph we have given μ_f the value p_7 = 140. As the range is $p_{m-3} \le \mu_f \le p_n$, μ_f can take all the pyramidal numbers lying between p_{m-3} and $p_n = p_{15} = 1240$ and hence the possible values of μ_f are 140, 204, 285, 385, 506, 650, 819, 1015, and 1240.

Theorem 2.11: All Stars $K_{1,n}$ are Edge Magic Pyramidal for $n \ge 3$ with the Magic constants μ_f ranging from $2n+3 \le \mu_f \le p_n$.

Proof: Let v_o be the root vertex of the Star $K_{1,n}$. Let v_i, i = 1 to n be the pendant vertices and e_i, i = 1 to n be the edges.

Define $f(v_0) = 1$

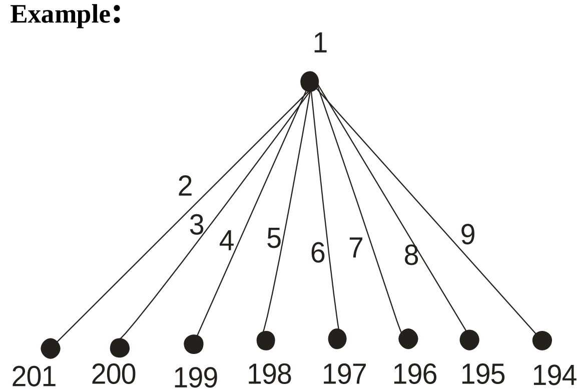
$f(v_1) = \mu_f - 3$

$f(v_i) = f(v_{i-1}) - 1$ for $2 \le i \le n$

$f(e_i) = i + 1$ for $1 \le i \le n$

By the above labeling all Stars $K_{1,n}$ are Edge Magic Pyramidal for $n \ge 3$.

Example:



Edge Magic Pyramidal labeling of $K_{1,8}$ (μ_f = 204)

Remark 2.12: For $K_{1,8}$ in this example we have given μ_f the value p_8 = 204, As 2n+3 $\leq \mu_f \leq p_n$, μ_f can take all the pyramidal numbers lying between 2n+3= 19 and $p_n = p_8 =$ 204 and hence the possible values of λ_f are 30, 55, 91, 140, 204.

III. Edge Magic Strength and Shift labelings

Definition 3.1: The Edge Magic Strength m(G) of a graph G is defined as the minimum of μ_f, where the minimum is taken over all Edge Magic Pyramidal labelings of G. Analogous to the minimum magic strength, the maximum magic strength M(G) is defined as the maximum of all μ_f.

Definition 3.2: An Edge Magic Pyramidal graph G is said to be Strong edge magic if m(G)=M(G), Ideal edge magic if M(G) – m(G) < p_q, Weak edge magic if M(G) – m(G) > p_q where p_q is the qth Pyramidal number.

Lemma 3.3: The Stars $K_{1,n}$ are Strong edge magic pyramidal for n = 3 and Ideal edge magic pyramidal for all n ≥ 4.

Proof: The Edge magic constants for all Stars μ_f range from 2n+3 $\leq \mu_f \leq p_n$. When n= 3 we have 9 $\leq \lambda_f \leq p_3$ = 14. Under this range 14 is the only Pyramidal number. Therefore m(G) = M(G) = 14. Hence $K_{1,n}$ is Strong edge magic pyramidal for n = 3. For all n ≥ 4 , M(G) = p_n and m(G) is approximately equivalent to 2n+3. Therefore we have M(G)–$m(G) = p_n - (2n+3) < p_n$ for any positive integer n ≥ 4. Hence the Stars $K_{1,n}$ are Ideal edge magic pyramidal for all n ≥ 4.

Lemma 3.4: The Peterson graph is an Ideal Edge Magic Pyramidal graph.

Proof: The Edge magic constants of a Peterson graph range from $p_{m-3} \leq \mu_f \leq p_n$ where m, n are the number of vertices and edges in the graph. Clearly $p_n - p_{m-3} < p_n$ for any m,n. In particular when m = 10 and n = 15 we have, $p_7 \leq \mu_f \leq p_{15}$. Hence 140 ≤ $\mu_f \leq$ 1240. Now M(G) = 1240 and m(G) = 140. M(G) – m(G) = 1100 < p_{15} = 1240 which implies that Peterson graph is Ideal edge magic Pyramidal.

Remark 3.5: All Complete bipartitite graphs $K_{m,n}$ are Weak edge magic pyramidal. For $K_{m,n}$, μ_f range from $p_{m+2} \leq \mu_f \leq p_{m^2+i}$ for m= n where i takes the values 0,2,4,6… for each m ranging from 2,3,4…. and for m $\neq n$, μ_f approximately varies from $p_{m+2} \leq \mu_f \leq p_{mn+m\sim n}$. Obviously we have $p_{m^2+i} - p_{m+2} > p_{mn}$, $p_{mn+m\sim n} - p_{m+2} > p_{mn}$ for any positive integers m and n which implies that $K_{m,n}$ are Weak edge magic pyramidal.

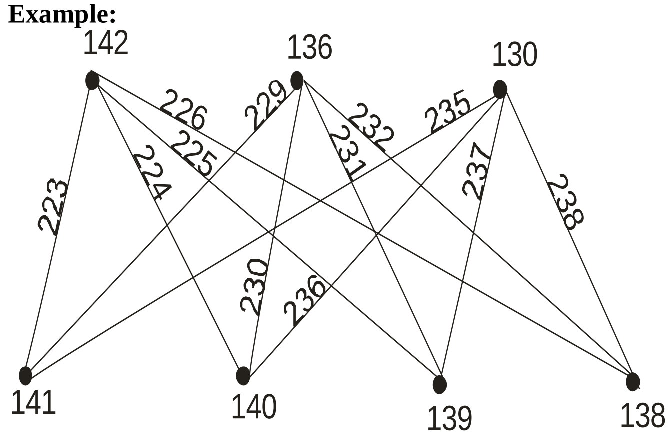
Definition 3.6: The process of Shifting an Edge magic pyramidal graph with a particular magic constant say K to any other magic constant K' such that either $K' > K$ or $K' < K$ is termed as Edge magic Shift labeling.

Definition 3.7: The Upper Edge magic Shift labeling of an Edge magic pyramidal graph is the process of shifting the graph with a particular magic constant K to any other magic constant . K' such that $K' > K$. Let Δ denote the Edge magic forward difference operator defined by $\Delta = K' - K$. By replacing all the edge labels in the primal graph by $f(e_i) + \Delta$ we get the Upper Edge magic Shift labeling.

Definition 3.8: The Lower Edge magic Shift labeling of an Edge magic pyramidal graph is the process of shifting the graph with a particular magic constant K to any other magic constant. K' such that $K' < K$. Let ∇ denote the Edge magic backward difference operator defined by $\nabla = K - K'$. Let S denote the set of vertex labels and T denote the set of edge labels. Let α_i, i = 1,2… denote the elements of S and β_j, j = 1,2… denote the elements of T. If each $\alpha_i > \nabla$ then in the primal graph replace the vertex labels $f(v_i)$ by $f(v_i) - \nabla$ for all i such that no two adjacent vertices undergo the operation. If each $\beta_j > \nabla$, then in the primal graph replace $f(e_i)$ by $f(e_i) - \nabla$ which in turn yields a Lower Edge magic Shift labeling.

Remark 3.9: The graph $K_{3,4}$ in Theorem 2.7 with magic constant μ_f = K = 285 has been shifted to an Edge magic pyramidal graph with a higher magic constant K' = 506 as follows: Let $\Delta = K' - K$ = 506 – 285 = 221. Now, by replacing all the edge labels in the primal graph by $f(e_i) +$ $\Delta = f(e_i) +$ 221 we get the Upper Edge magic Shift labeling.

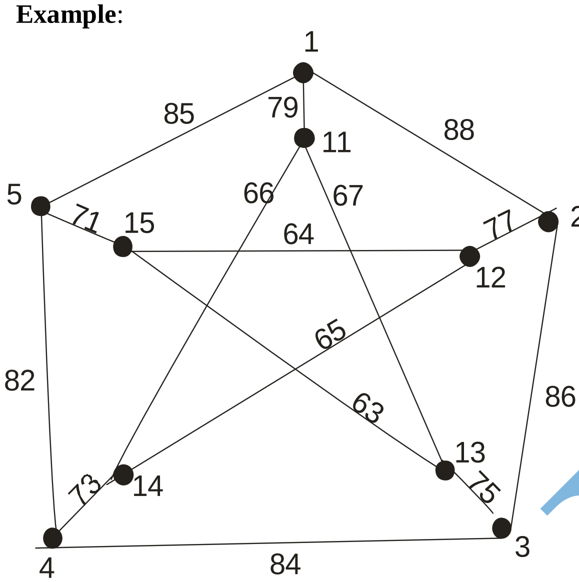
Example:



Upper Edge Magic Shift labeling of $K_{3,4}$ ($\mu_f = K' = 506$)

Remark 3.10: The Peterson graph in Theorem 2.9 with magic constant μ_f = K = 140 has been shifted to an Edge magic pyramidal graph with a lower magic constant K' = 91 as follows: Let $\nabla = K - K'$ = 140– 91 = 49. Let S denote the set of vertex labels and T denote the set of edge labels. Let α_i, i = 1,2… denote the elements of S and β_j, j = 1,2… denote the elements of T. As $\beta_j > \nabla$, in the primal graph replacing $f(e_i)$ by $f(e_i) - \nabla$ we get the Lower Edge magic Shift labeling

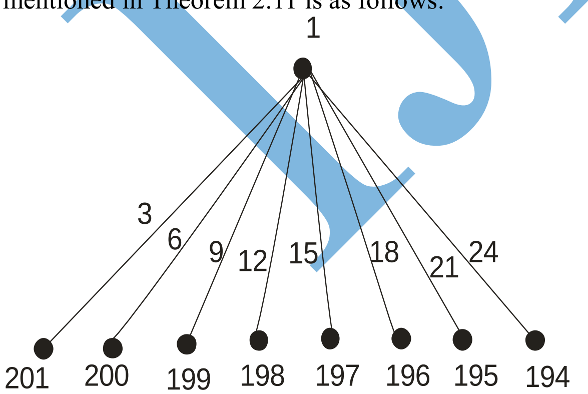
Example:



Lower Edge Magic Shift labeling of Peterson graph (μ_f = K' = 91)

Definition 3.11: An Edge Antipyramidal labeling is a labeling derived from an Edge magic pyramidal graph with a particular Edge magic constant μ_f by adding the odd numbers 1,3,5… successively with either the vertex labels or the edge labels which are of the least range such that the weight of all the edges f(u) + f(v) +f(uv) where uv $\in E(G)$ are distinct and they are not Pyramidal numbers.

Example: The Antipyramidal labeling of the Star graph mentioned in Theorem 2.11 is as follows:



In this labeling the distinct Edge weights are 205,207,209,211,213,215,217, 219 and they are not Pyramidal numbers. They form an Arithmetic Progression with the first term as a = 205 and common difference d = 2.

IV. CONCLUSION

Analogous to Edge magic Pyramidal graphs Vertex magic Pyramidal graphs are also introduced. Since every edge is incident with two vertices most of the graphs satisfy the condition of Edge magic Pyramidal labeling. But it is interesting to investigate graphs that are Edge magic Pyramidal but not Vertex magic Pyramidal. Those graphs can be termed as Partially magic Pyramidal graphs. This work has brought Pyramidal numbers into existence. As the difference between any two pyramidal numbers is a perfect square and the difference is sufficiently large, Pyramidal numbers can be used in distance labelings as frequencies of the transmitters for better transmission.

REFERENCES:

[1] M. Apostal, **Introduction to Analytic Number Theory,** Narosa Publishing House, Second edition, 1991.

[2] J.A. Bondy and U.S.R. Murty, **Graph Theory with applications,** Macmillan press,London(1976).

[3] J.A. Gallian, **A Dynamic Survey of Graph Labeling,** The Electronic Journal of Combinatorics (2014).

[4] F. Harary, **Graph Theory** Addison – Wesley, Reading Mars., (1968).

[5]W.D. Wallis,Edy T. Baskoro and Mirka Miller Slamin, **Edge Magic Total Labelings,** The Australasian Journal of Combinatorics. 22(2000).pp.177-190.

