On β* - closed and β* - open maps in Topological Spaces

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Abstract- **We investigated a new class of β*- open and β*- closed maps in topological spaces and study some of its basic properties and relations among them. It is shown that the composition of β*- closed maps need not be β*- closed .The applications of these maps in some topological spaces are also studied.**

Keywords: β* -open maps and β*- closed maps.

I. INTRODUCTION

In 1960, Levine . N [7] introduced strong continuity in topological spaces. Abd El-Monsef et al. [1] introduced the notion of β-open sets and β-continuity in topological spaces. Semi-open sets, preopen sets, α-sets, and β-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of generalizations of continuity. In 1982, Mashhour et. al. [10] introduced preopen sets and pre-continuity in topology. Levine [5] introduced the class of generalized closed (g-closed) sets in topological spaces. In this paper we introduce and investigate a new class of functions called β* - open and β* - closed maps and their relations with various maps.

II. PRELIMINARIES

Throughout this paper (X, τ), (Y, σ) and (Z, η) or X , Y , Z represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure and the interior of A respectively. The power set of X is denoted by P(X).

Definition 2.1: A subset A of a topological space (X, τ) is called β *- closed Set if Int*(Cl(Int*(A))) ⊆ A.

Definition 2.2: A subset A of a topological space (X, τ) is called β *- open Set if X \ A is β *- closed Set.

Definition 2.3: A function f: (X, τ) ⟶ (Y, σ) is called a β *- continuous if f^{-1}(O) is a β * - open set of (X, τ) for every open set O of (Y, σ).

Definition 2.4: A function f: (X, τ) ⟶ (Y, σ) is called a g - continuous if f^{-1}(O) is a g -open set of (X, τ) for every open set O of (Y, σ).

Definition 2.5: A function f: (X, τ) ⟶ (Y, σ) is called a perfectly continuous if f^{-1}(O) is both open and closed in (X, τ) for every open set O in (Y, σ).

Definition 2.6: A map f: (X, τ) ⟶ (Y, σ) is called a g-closed if f(O) is g-closed in (Y, σ) for every closed set O in (X, τ).

Definition 2.7: A Topological space X is said to be β * - $T_{1/2}$ space if every β * - open set of X is open in X.

Theorem 2.8:

(i) Every open set is β *- open and every closed set is β *- closed set

(ii) Every β -open set is β *- open and every β -closed set is β *- closed.

(iii) Every g-open set is β *-open and every g-closed set is β *-closed.

III. β * - Open maps and β * - Closed maps

Definition 5.1: A map f: (X, τ) ⟶ (Y, σ) is called a β *- open if image of each open set in X is β * - open in Y.

Definition 5.2: A map f: (X, τ) ⟶ (Y, σ) is called a β *-closed if image of each closed set in X is β * -closed in Y

Theorem 5.3: Every closed map is β *- closed map.

Proof: The proof follows from the definitions and fact that every closed set is β * - closed.

Remark 5.4: The converse of the above theorem need not be true.

Example 5.5: Let X = Y= {a, b, c}, τ = {ϕ,{a},{a, b},{a, c}, X}, τ^c = { ϕ,{b},{c},{b, c}, X} and σ = {ϕ, {a, b},Y}, σ^c ={ ϕ,{c}, Y}, β * C(Y, σ) = { ϕ, {a},{ b},{c},{a, c},{b, c}, Y } Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = a , f(b) = c, f(c) = b. clearly, f is β * - closed but not closed as the image of closed set f(c) ={b} ,f(b, c) ={b, c} in X is not closed in Y.

Theorem 5.6: Every g-closed map is β * - closed.

Proof: Let O be a closed set in X. Since f is g-closed map, f(O) is g-closed in Y. By [8] f(O) is β * -closed in Y. Therefore, f is β * - closed map.

Remark 5.7: The converse of the above theorem need not be true.

Example 5.8: Let X = Y= {a, b, c}, τ = {ϕ,{a},{a, b},{a, c}, X}, τ^c = { ϕ,{b},{c},{b, c}, X} and σ = {ϕ, {a},Y}, σ^c ={ ϕ,{b, c}, Y}, β * C(Y, σ) = { ϕ, {a},{ b},{c},{a, b}, {a, c},{b, c}, Y }, g C(Y, σ) = { ϕ, { b},{c},{a, b}, {a, c},{b, c}, Y } Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = b , f(b) = a, f(c) = c. clearly, f is β * - closed but not g - closed as the image of closed set {b} in X is {a} which is not g - closed set in Y.

Theorem 5.9: Every β - closed map is β * - closed.

Proof: The proof follows from the definition.

Remark 5.10: The converse of the above theorem need not be true.

Example 5.11: Let X = Y= {a, b, c, d }, τ = {ϕ,{a, b},{a, b, c}, X}, τ^c = { ϕ,{d},{c, d}, X}and σ = {ϕ,{a},{a, b, c},Y} , β *C(Y, σ) = {ϕ, {a},{b},{c},{d},{a, b},{a, c},{a, d},{b, c},{b, d},{c, d}, {a, b, c},{a, b, d},{a, c, d},{b, c, d},Y}, β C(Y, σ) ={ϕ,{a},{b},{c},{d},{a, c},{a, d},{b, c},{b, d},{c, d},{a, c, d},{b, c, d},Y} . Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = d , f(b) = c, f(c) = b , f(d)= a. Clearly, f is β * - closed but not β - closed as the image of closed set {c, d} in X is {a, b} which is not in β -closed set in Y.

Remark 5.12: The composition of two β * - closed maps need not be β * - closed in general as shown in the following example.

Example 5.13: Consider X = Y = Z = {a, b, c}, τ = {ϕ, {a}, X}, τ^c = {ϕ, {b, c}, X} and σ = {ϕ,{a, b},Y}, σ^c ={ ϕ,{c}, Y} , η = {ϕ,{a},{a, b},{a, c},Z}, β *C(Z, η) ={ϕ,{b},{c},{b, c}}, β *C(Y, σ) ={ϕ,{a},{b},{c},{a, c}, {b, c}, Y}. Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = b , f(b) = a, f(c) = c . Clearly, f is β *- closed. Consider the map g: (Y, σ) ⟶(Z, η) defined g(a) = a, g(b) = b, g(c) = c , clearly g is β * - closed. But g ∘f : (X, τ) ⟶(Z, η) is not a β * - closed, g ∘f ({b, c})= g (f{b, c}) = g ({a, c}) = {a, c} which is not a β * - closed in Z.

Theorem 5.14: A map f: (X, τ) ⟶ (Y, σ) is β * - closed if and only if β * cl (f(A)) ⊆ f(cl(A)) for each set A in X.

Proof: Suppose that f is a β *- closed map. Since for each set A in X, cl(A) is closed set in X, then f(cl(A)) is a β *- closed set in Y. Since, f(A) ⊆ f(cl(A)), then β * cl (f(A)) ⊆ f(cl(A)) Conversely, suppose A is a closed set in X. Since β * cl (f(A)) is the smallest β *- closed set containing f(A), then f(A) ⊆ β * cl (f(A)) ⊆ f(cl(A)) = f(A). Thus, f(A) = β * cl (f(A)). Hence, f (A) is a β *- closed set in Y. Therefore, f is a β * - closed map.

Theorem 5.15: If f: (X, τ) ⟶ (Y, σ) is closed map and g: (Y, σ) ⟶ (Z, η) is β *- closed, then the composition g ∘f : X ⟶ Z is β * - closed map.

Proof: Let O be any closed set in X. Since f is closed map. f(O) is closed set in Y. Since, g is β *- closed map, g(f(O)) is β *- closed in Z which implies g ∘f ({O}) = g(f{O}) is β *- closed and hence , g ∘f is β *- closed.

Remark 5.16: If f: (X, τ) ⟶ (Y, σ) is β * - closed map and g: (Y, σ) ⟶ (Z, η) is closed, then the composition g ∘f : X ⟶ Z is not β *- closed map as shown in the following example.

Example 5.17: Consider X = Y = Z = {a, b, c}, τ = {ϕ,{a}, X}, τ^c = {ϕ, {b, c}, X} and σ = {ϕ,{a, b},Y},η ={ϕ,{a},{a, b},{a, c}, Z}, η^c = {ϕ, {b},{c},{b, c}, Z}. β *C(Y, σ)= {ϕ , {a} , {b} , {c} ,{a, c} ,{b, c} ,Y}, β * C (Z, η) = {ϕ ,{b}, {c},{b, c}, Z}. Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = b , f(b) = a, f(c) =c . Clearly, f is β * - closed. Consider the map g: (Y, σ) ⟶ (Z, η) defined g(a) = a, g(b) = b, g(c) = c ,clearly g is closed. But g ∘f : (X, τ) ⟶ (Z, η) is not a β * - closed, g ∘f ({b, c}) = g (f{b, c}) = g ({a, c}) = {a, c} which is not a β * - closed in Z.

Theorem 5.18: Let (X, τ) , (Z, η) be topological spaces and (Y, σ) be topological spaces where every β * - closed subset is closed. Then the composition g ∘f : (X, τ) ⟶ (Z, η) of the β * -closed , f: (X, τ) ⟶ (Y, σ) and g: (Y, σ) ⟶ (Z, η) is β * - closed.

Proof: Let O be a closed set in X . Since, f is β * - closed, f(O) is β *- closed in Y. By hypothesis, f(O) is closed .Since g is β * - closed, g(f{O}) is β * - closed in Z and g(f{O}) = g ∘f (O). Therefore, g ∘f is β * - closed.

Theorem 5.19: If f: (X, τ) ⟶ (Y, σ) is g - closed map and g: (Y, σ) ⟶ (Z, η) is β * - closed and (Y, σ) is T $_{1/2}$ spaces . Then the composition g ∘f : (X, τ) ⟶ (Z, η) is β * - closed map.

Proof: Let O be a closed set in (X, τ). Since f is g – closed , f(O) is g – closed in (Y, σ) and g is β * - closed which implies g(f(O)) is β * - closed in Z and g(f(O)) = g ∘f (O). Therefore, g ∘f is β * - closed.

Theorem 5.20: Let f: (X, τ) ⟶ (Y, σ) and g: (Y, σ) ⟶ (Z, η) be two mappings such that their composition g ∘f : (X, τ) ⟶ (Z, η) be β * - closed mapping. Then the following statements are true.

1. If f is continuous and surjective, then g is β * - closed.
2. If g is β * - irresolute and injective, then f is β * - closed.
3. If f is g - continuous , surjective and (X, τ) is a T 1/2 spaces, then g is β * - closed.
4. If g is strongly β * - continuous and injective, then f is β * - closed.

Proof: 1. Let O be a closed set in (Y, σ) . Since , f is continuous, f-1(O) is closed in (X, τ) . Since, g ∘f is β * - closed which implies g ∘f (f^{-1} (O)) is β * - closed in (Z, η). That is g(O) is β * - closed in (Z, η), since f is surjective. Therefore, g is β * - closed.

2. Let O be a closed set in (X, τ). Since g ∘f is β * - closed, g ∘f (O) is β * - closed in (Z, η), Since g is β * - irresolute, g^{-1} (g ∘f (O)) is β * - closed in (Y, σ). That is f(O) is β * - closed in (Y, σ). Since f is injective. Therefore, f is β * - closed.

3. Let O be a closed set of (Y, σ). Since, f is g- continuous , f^{-1} (O) is g – closed in (X, τ) and (X, τ) is a $T_{1/2}$ spaces, f^{-1} (O) is closed in (X, τ) . Since , g ∘f is β * - closed which implies , g ∘f (f^{-1} (O)) is β * - closed in (Z, η). That is g(O) is β * - closed in (Z, η) , since f is surjective. Therefore, g is β * - closed.

4. Let O be a closed set of (X, τ).Since, g ∘ f is β * - closed which implies , g ∘f (O) is β * - closed in (Z, η). Since, g is strongly β *- continuous, g^{-1} (g ∘f (O)) is closed in (Y, σ).That is f(O) is closed in (Y, σ). Since g is injective, f is β * - closed.

Theorem 5.21: A map f: (X, τ) ⟶ (Y, σ) is β *- open if and only if f (int(A)) $\subseteq$ β * int (f(A)) for each set A in X.

Proof: Suppose that f is a β * - open map. Since int (A) $\subseteq$ A, then f(int (A)) $\subseteq$ f(A). By hypothesis, f(int (A)) is a β *-open and β * int (f(A)) is the largest β *- open set contained in f(A).Hence f(int(A)) $\subseteq$ β * int (f(A)).Conversely, suppose A is an open set in X. Then f(int(A)) $\subseteq$ β * int (f(A)) . Since int (A) = A, then f(A) $\subseteq$ β * int (f(A)). Therefore, f(A) is a β * - open set in (Y, σ) and f is β *-open map.

Theorem 5.22: Let (X, τ) , (Y, σ) and (Z , η) be three topologies spaces f: (X, τ) ⟶ (Y, σ) and g: (Y, σ) ⟶ (Z , η) be two maps. Then

1. If (g ∘ f) is β *- open and f is continuous, then g is β *-open.

2. If (g ∘ f) is open and g is β * -continuous, then f is β *-open map.

Proof:

1. Let A be an open set in Y .Then, f^{-1} (A) is an open set in X. Since (g ∘ f) is β *- open map, then (g ∘ f) (f^{-1} (A)) = g(f((f^{-1} (A)) = g(A) is β *- open set in Z. Therefore, g is a β *- open map.

2. Let A be an open set in X. Then, g(f(A)) is an open set in Z. Therefore, g^{-1} (g(f(A))) = f(A) is a β * - open set in Y . Hence, f is a β *- open map.

Theorem 5.23: Let f: (X, τ) ⟶ (Y, σ) be a bijective map. Then the following are equivalent:

(1) f is a β * - open map.

(2) f is a β * - closed map.

(3) f^{-1} is a β *- continuous map.

Proof:

(1) $\Rightarrow$ (2) Suppose O is a closed set in X. Then X\O is an open set in X and by (1) f(X\O) is a β *- open in Y. Since, f is bijective, then f(X\O) = Y\ f(O). Hence, f(O) is a β *- closed in Y. Therefore, f is a β *- closed map.

(2) $\Rightarrow$ (3) Let f is a β *- closed map and O be closed set in X. Since, f is bijective then $(f^{-1})^{-1}$ (O) = f(O) which is a β *- closed set in Y. Therefore, f is a β *-continuous map.

(3) $\Rightarrow$ (1) Let O be an open set in X. Since, f^{-1} is a β *-continuous map then $(f^{-1})^{-1}$ (O) = f(O) is a β *-open set in Y. Hence, f is β *- open map.

Theorem 5.24: A map f: (X, τ) ⟶ (Y, σ) is β *- open if and only if for any subset O of (Y, σ) and any closed set of (X, τ) containing f^{-1} (O), there exists a β *- closed set A of (Y, σ) containing O such that f^{-1} (A) $\subset$ F.

Proof: Suppose f is β *- open. Let O $\subset$ Y and F be a closed set of (X, τ) such that f^{-1} (O) $\subset$ F. Now X- F is an open set in (X, τ). Since f is β *- open map, f (X – F) is β *- open set in (Y, σ). Then, A = Y - f (X – F) is a β *- closed set in (Y, σ). Note that f^{-1} (O) $\subset$ F implies O $\subset$ A and f^{-1} (A) = X – f^{-1} (X – F) $\subset$ X - (X – F) = F. That is , f^{-1} (A) $\subset$ F. Conversely, let B be an open set of (X, τ) . Then, f^{-1} $((f(B))^c)$ $\subset$ B^c and B^c is a closed set in (X, τ). By hypothesis, there exists a β *- closed set A of (Y, σ) such that $(f(B))^c$ $\subset$ A and f^{-1} (A) $\subset$ B^c and so B $\subset$ $(f^{-1}(A))^c$. Hence, A^c $\subset$ f(B) $\subset$ f $((f^{-1}(A)))^c$ which implies f(B) = A^c . Since, Ac is a β *- open. f (B) is β *- open in (Y, σ) and therefore f is β *-open map.

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