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A Finite Population Retrial Inventory System with Service Facility and Phase Type Server Vacation Times

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***Abstract*—In this article, we consider a continuous review inventory system with demands are generated by a finite number of homogeneous population. The maximum storage capacity is S and the demand time points form a Quasi random input. The inventory is delivered after some random time due to service on it. Here we assume exponential service time. The inventory is replenished according to (s, S) ordering policy and the lead times are assumed to follow an exponential distribution. Whenever the inventory level reaches zero the server goes for vacation. The duration of server vacation is distributed as phase type. The demands that occur during stock out period and/or during the server vacation period is permitted enter into the orbit. These orbiting demands retry for their demand after a random time, which is distributed as exponential. The joint probability distribution of the inventory level and the number of demands in orbit are obtained in the steady state case. Various system performance measures and the long run total expected cost rate function are derived.**

Subject Classification: 90B05, 60J27

***Keywords*— Continuous review Inventory Systems, (s, S) Policy, Finite population, Service facility, Server vacation, Retrial demands, Phase type distributio**n.

I. INTRODUCTION

In all works reported in inventory prior to 1993, it was assumed that the time required to serve the items to the customer is negligible. Berman et al. [2] were the first to attempt to introduce positive service time in inventory, where it was assumed that service time is a constant. Later, Berman and Kim [3] extended this result to random service time.
However, in Queueing systems with server vacations have been widely studied in different contexts in the literature(see for example Takagi [7]). Continuous review inventory system with server vacation has drawn little attention[4].
Here we deal with a branch of the inventory theory which is characterized by the following feature: a demand cannot receive the item (due to empty stock/server's vacation) leaves the system but after some random time returns to the system again. As a consequence, repeated attempts for an item from the orbit(i.e., the retrial pool of unsatisfied demands) are superimposed on the ordinary stream of arrivals of first attempts. The pioneering studies of retrial system [1] present the concept of retrial time as an alternative to the classical models of telephone systems, queues with losses, that do not take repeated calls into account.
In this paper, we address a continuous review perishable inventory system with a finite number of homogeneous sources of demands. The inventory is delivered to the customer after some random time due to service on it. The server takes a vacation each time whenever the inventory level becomes zero. When he returns from vacation, if the inventory level is still zero, he begins another vacation; if the stock replenished he terminates his vacation, and he is ready to serve any arriving demands. During the vacation period, any arriving primary demands enter the orbit for retry after a random time. We assume the server's vacation time as phase type distribution. To get away from Poisson/exponential models, Neuts [5] developed the
theory of PH-distributions and related point processes. In stochastic modeling, PH-distributions lend themselves naturally to algorithmic implementations. They have very nice closure properties and a related matrix formulation that make them attractive and more effective for use in practice. In this paper we assume the server's vacation time follows phase type distribution.
The rest of the paper is organized as follows. In Section 2, we describe the problem and in the next section analyse the mathematical model of the problem under study. The steady-state analysis of the model is presented in section 4 and some key system performance measures and the long run total expected cost rate function are derived in Section 5.
Notation:

- A(i,j) : entry at $(i,j)^{th}$ position of A
- 0 :Zero vector of approprite dimension
- e :a column vector of 1's of appropriate dimension
- I_n : Identity matrix of order n
- $\delta_{i,j}$:Kronecker delta function

II. Problem Information:

Consider an inventory system with a maximum capacity of S for stocking units at a service facility. The customers arrive to the service station from a finite population of size M . Each customer is either free or in the orbit at any time. The input process is characterized by the fact that each free customer generates demands independently and with the same exponentially distributed inter-arrival time of rate λ. This particular type of finite-source input is often called quasi random input. The items are delivered to the customers after performing the service on the items. The service time is a continuous random variable and it follows exponential distribution with parameter µ. The operating policy is (s, S) ordering policy. According to this policy, when the on hand inventory level reaches the prefixed level s, he places an order for Q(= S − s) items. The lead time is exponentially distributed with mean rate γ(> 0). When the inventory level drops to zero, the server leaves for vacation. If the server finds an empty stock at the end of a vacation, he takes another vacation immediately. The server terminates his vacation only when he finds the positive inventory level. The duration of server vacation follows phase type distribution with representation (θ, T). Thus, the mean retrial time is θ(−T)−1 e. We denote the exit rate vector by t = −T e. Demands that occur during the server vacation period and or server busy period enter into the orbit. These orbiting demands compete for their demands according to an exponential distribution. We consider the linear retrial policy. It means, the time intervals between successive repeated attempts are exponentially distributed random variable with parameter jα + (1 − δj,0)β, when the orbit size is j. We also assume that the inter-arrival times between the primary demands, service time, lead time, retrial times and server vacation time are mutually independent random variables.

III. Analysis

Let I(t),O(t) and V(t) ,respectively ,denote the inventory level,number of customer in the orbit and the phase of the vacation process at time t.Define the variable,

$$S_e(t) = \begin{cases} 0, & \textit{server is on vacation} \\ 1, & \textit{server is idle} \\ 2, & \textit{server is busy} \end{cases}$$

From the assumption made on the input and output processes, it may be verified that the stochastic process $\{I(t), O(t), S_e(t), V(t), t \geq 0\}$ is a Markov process with state space E,which is define as,

$$E = \bigcup_{i=0}^{S} a(i) = \bigcup_{i=0}^{S} \bigcup_{j=0}^{M} a(i,j)$$

$$a(0,j) = \{0, j, 0, l\} / j = 0,1,2.......M., l = 1,2,3.....n$$

$$a(i,j) = \{(i,j,k) i = 0,1,2.......Q-1, j = o,1,2,......M-1, k = 1,2\}$$

$$a(i,j) = \{(i,j,k), i = 0,1,2.......Q-1, j = M, k = 1\}$$

$$a(Q,j) = \{(Q,j,0,l)(Q,j,1),(Q,j,2)\} / j = 0,1,2.......M-1, l = 1,2,3.....n\}$$

$$a(Q,M) = \{((Q,M,0,l)(Q,M,1)) \mid l = 1,2......n$$

$$a(i,j) = \{(i,j,k) \mid i = Q+1, Q+2,...........S, j = 0,1,2,....M-1, K = 1,2\}$$

$$a(i,j) = \{(i,j,k) \mid i = Q+1, Q+2,...........S, j = M, K = 1\}$$

Then the infintesimal generator P can be conveniently expressed in block partitioned matrix with entries,

$$P = \begin{pmatrix} A_0 & & & B_0 & & \\ C_1 & A_1 & & & B_1 & \\ & & & & & \\ & C_s & A_s & & & B_s \\ & & & & C_s & A_s \end{pmatrix}$$

For i=1,2,3......S,

$$[A_i]_{jk} = \begin{cases} D_{ij} & k = j; j = 0,1,2,...........M \\ E_{ij} & k = j+1; j = 0,1,2,.......M-1 \\ F_{ij} & k = j-1; j = 1,2,3..........M \\ 0, & \textit{othewies} \end{cases}$$

For i=1,2,3......s,

$$[B_i]_{jk} = \begin{cases} G_{ij} & k = j; j = 0,1,2,3.........M \\ 0, & \textit{othewies} \end{cases}$$

For i=1,2,3......S,

$$[C_i]_{jk} = \begin{cases} H_{ij} & k = j; j = 0,1,2,3.........M-1 \\ 0, & \textit{othewies} \end{cases}$$

For j=1,2,3......M,

$$E_{oj} = \begin{matrix} & 0 \\ 0 & ((M-j)\lambda I_n) \end{matrix}$$

$$D_{oj} = \begin{matrix} & 0 \\ 0 & ((T + t \otimes \theta - ((M-j)\lambda + \gamma) I_n) \end{matrix}$$

For j=1,2,3......s,

$$D_{ij} = \begin{matrix} & 1 & 2 \\ 2 & \begin{pmatrix} -(-M-j)\lambda - (j\alpha + \beta) - \gamma & (M-j)\lambda \\ & -(-M-1-j)\lambda - i\mu - \gamma \end{pmatrix} \end{matrix}$$

$$D_{iM} = \begin{matrix} & 1 \\ 1 & (-(j\alpha+\beta)-\gamma) \end{matrix}$$

$$D_{Qj} = \begin{matrix} & 0 & 1 & 2 \\ 0 & T & t & \\ 1 & & -(M-j)\lambda-(j\alpha+\beta) & (M-j)\lambda \\ 2 & & & -(M-j-1)\lambda-i\mu \end{matrix}$$

$$D_{QM}^{00} = \begin{matrix} & 0 \\ 0 & (-\beta-\psi_{ij}) \end{matrix}$$

$$E_{ij} = \begin{matrix} & 1 & 2 \\ 1 & & \\ 2 & & (M-1-j)\lambda \end{matrix}$$

$$F_{ij} = \begin{matrix} & 1 & 2 \\ 1 & & (j\alpha+\beta) \\ 2 & & \end{matrix}$$

For i = s + 1, s + 2, . . . , Q − 1 and i = Q + 1, Q + 2, . . . , S

$$E_{ij} = \begin{matrix} & 1 & 2 \\ 1 & & \\ 2 & & (M-1-j)\lambda \end{matrix}$$

$$F_{ij} = \begin{matrix} & 1 & 2 \\ 1 & & (j\alpha+\beta) \\ 2 & & \end{matrix}$$

$$F_{iM} = \begin{matrix} & 1 & 2 \\ 1 & & \\ 2 & & (j\alpha+\beta) \end{matrix}$$

$$D_{ij} = \begin{matrix} & 1 & 2 \\ 2 & -(-M-j)\lambda-(j\alpha+\beta)-\gamma & (M-j)\lambda \\ & & -(-M-1-j)\lambda-i\mu-\gamma \end{matrix}$$

$$E_{Qj} = \begin{matrix} & 0 & 1 & 2 \\ 0 & & & \\ 1 & & & \\ 2 & & & (M-1-j)\lambda \end{matrix}$$

$$F_{Qj} = \begin{matrix} & 0 & 1 & 2 \\ 0 & & & \\ 1 & & & j\alpha+\beta \\ 2 & & & \end{matrix}$$

$$F_{QM} = \begin{matrix} & 0 & 1 & 2 \\ 0 & & & \\ 1 & & & \\ 2 & & & M\alpha+\beta \end{matrix}$$

$$D_{Qj} = \begin{matrix} & 0 & 1 & 2 \\ 0 & T & t & \\ 1 & & -(M-j)\lambda-(j\alpha+\beta) & (M-j)\lambda \\ 2 & & & -i\mu-(M-1-j)\lambda \end{matrix}$$

$$D_{QM} = \begin{matrix} & 0 & 1 \\ 0 & T & t \\ 1 & & -(j\alpha+\beta) \end{matrix}$$

The dimension of the matrices is given by the following table,

Matrix	Dimension	Matrix	Dimension
A_0	(n(M+1),n(M+1))	B_0	(n(M+1),2M+1+n(M+1))
A_Q	(2M+1+n(M+1),2M+1+n(M+1))	B_1	(2M+1,2M+1)
A_{Q+1}	(2M+1,2M+1)	C_1	(n(M+1),2M+1+n(M+1))
For i=1,2,3.... Q-1 &i=Q+2,Q+3,......S	A_i->(2(M+1),2(M+1)) C_i->(2(M+1),2(M+1))	C_Q C_{Q+1}	(2M+1+n(M+1),2M+1) (2M+1,2M+1+n(M+1)

IV. STEADY STATE ANALYSIS

Since the state space E is finite and P is irreducible, the stationary probability vector Θ for the generator P exists and satisfies ΘP = 0, Θe = 1.

Using the structure of P the vector Θ can be represented by

$\Theta^{<<i>>} = (\Theta^{<i,0>}, \Theta^{<i,1>}, \dots, \Theta^{<i,M>})$, i = 0, 1, . . . , S

$\Theta^{<0,j>} = (\Theta^{(0,j,0)})$, j = 0, 1, . . . , M,

$\Theta^{<i,j>} = (\Theta^{(i,j,1)}, \Theta^{(i,j,2)})$, i = 1, 2, . . . , Q − 1, Q + 1, Q + 2, . . . , S, j = 0, 1, . . . , M,

$\Theta^{<Q,j>} = (\Theta^{(Q,j,0)}, \Theta^{(Q,j,1)}, \Theta^{(Q,j,2)})$, j = 0, 1, . . . , M,

$\Theta^{<i,M>} = (\Theta^{(i,M,1)})$, i = 1, 2, . . . , Q − 1, Q + 1, Q + 2, . . . , S,j=0,1,2..........M

$\Theta^{<Q,M>}=(\Theta^{(Q,M,0)},\Theta^{(Q,M,1)})$

The equation $\Theta P = 0$ yields the following set of equations :

$\Theta^{<<i+1>>} C_{i+1} + \Theta^{<<i>>} A_i = 0$,i=0,1,2,........Q-1,

$\Theta^{<<i+1>>} C_{i+1} + \Theta^{<<i>>} A_i + \Theta^{<<i-Q>>} B_0 = 0$,i=Q (1)

$\Theta^{<<i+1>>} C_{i+1} + \Theta^{<<i>>} A_i + \Theta^{<<i-Q>>} B_1 = 0$,i=Q+1..........S-1,

$\Theta^{<<i>>} A_i + \Theta^{<<i-Q>>} B_1 = 0$,i=S.

The equations (except (1)) can be recursively solved to get

$\Theta^{<<i>>} = \Theta^{<<Q>>} \Lambda_i$,i=0,1,2,..................S

Where

$$\wedge_i = \begin{cases} (-1)^{Q-i} C_Q A_{Q-1}^{-1} C_{Q-1} .... C_{i+1} A_i^{-1}, & i = 0,1,2,........Q-1 \\ I, & i = Q \\ (-1)^{2Q-i+1} \sum_{j=0}^{S-i} [(C_Q A_{Q-1}^{-1} C_{Q-1} .... C_{s+1-j} A_{S-j}^{-1}) B_1 A_{S-j}^{-1} X (C_{S-j} A_{S-j-1}^{-1} C_{S-j-1} .... C_{i+1} A_i^{-1})], & i = Q+1,.....S \end{cases}$$

and $\Theta^{<<Q>>}$ can be obtained by solving

$\Theta^{<<Q>>}$[

$$(-1)^Q \sum_{j=0}^{S-1} [(C_Q A_{Q-1}^{-1} C_{Q-1} .... C_{S+1-j} A_{S-j}^{-1}) B_1 A_{S-j}^{-1} X (C_{S-j} A_{S-j-1}^{-1} C_{S-j-1} ...... C_{Q+2} A_{Q+1}^{-1})] C_{Q+1} + A_Q + (-1)^Q C_Q A_{Q-1}^{-1} C_{Q-1} .... C_1 A_0^{-1} B_0] = 0$$

]

and

$\Theta^{<<Q>>}$[

$$\sum_{j=0}^{Q-1} [(-1)^{Q-i} (C_Q A_{Q-1}^{-1} C_{Q-1} .... C_{i+1} A_i^{-1}) + I + \sum_{i=Q+1}^{S} ((-1)^{2Q-i+1} \sum_{i=Q+1}^{S-i} [((C_Q A_{Q-1}^{-1} C_{Q-1} ...... C_{S+1-j} A_{S-j}^{-1})] C_{S-j} A_{S-j-1}^{-1} C_{S-j-1} .... C_{i+1} A_i^{-1}] e = 1.$$

]

V. System Performance Measures:

In this section ,we derive some system performence measures in the steaddy-state case.

5.1 Expected Inventory level:

Let $\Im_i$ denote the expected reorder level in the steady –state.Then ζ_i is given by

$$\Im_i = \sum_{i=1}^{S} i \Theta^{<<i>>} e$$

5.2 Expected reorder rate:

Let $\Im_r$ denote the expected reorder level in the steady –state. Then ζ_r is given by

$$\Im_r = \sum_{k=0}^{M-1} (s+1) \mu \Theta^{(s+1,j,2}$$

5.3 Expected number of customer in the orbit:

Let $\Im_o$ denote the expected reorder level in the steady –state.Then ζ_o is given by

$$\Im_o = \sum_{i=0}^{S} \sum_{j=1}^{M} j \Theta^{<<i,j>>} e$$

5.4 Mean number of departure :

The mean service time is defined as

$$\Im_{ser} = \sum_{i=1}^{S} \sum_{j=0}^{M-1} i \mu \Theta^{<<i,j,2>>} e$$

5.5 The fraction of time the server is on vacation

The fraction of time the server is on vacation is given by

$$\Im_{SV} = \sum_{i=0}^{M} [\Theta^{<<0,j,0>>} e + \Theta^{<<Q,k,0>>}]$$

5.6 Fraction of successful rate of retrials:

The fraction of successful rate of retrials is given by

$$\Im_{FR} = \frac{The\ Successful\ rate\ of\ retrial}{The\ over\ all\ rate\ of\ retrial}$$

$$= \frac{\sum_{i=1}^{S} \sum_{j=1}^{M} (j\alpha + \beta) \Theta^{(i,k,1)}}{\sum_{i=1}^{S} \sum_{j=1}^{N} (j\alpha + \beta) \Theta^{<<i,j>>} e}$$

VI. Cost Analysis

The long –run expected cost rate for this model is defined to be

$$TC(S,s) = c_1 \sum_{i=1}^{S} i \Theta^{<<i>>} e + c_2 \sum_{k=0}^{M-1} (s+1) \mu \Theta^{(s+1,j,2)} + c_3 \sum_{i=1}^{S} \sum_{j=0}^{M-1} i \mu \Theta$$

Where

C_1 : The inventory carrying cost per unit item per unit time

C_2: Setup cost per order

C_3: cost due to service/unit /unit time

C_4: Waiting cost of a customer in the orbit per unit time

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