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Pointwise convergence of prolate spheriodal wavelet expansion in $L^2(R)$ space

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Abstract: **In this paper we shall study the point wise convergence of prolate spheroidal wavelet expansions in L2 (R). Here the sinc function is replaced by one based on prolate spheroidal wave functions (PSWF's) which have much better time localization than the sinc function. The new wavelets preserve the high energy concentration in both the time and frequency domain inherited from PSWF's**

Keywords and Phrases: **Orthonormal basis, Prolate spheroidal wavelets , L2(R) spaces and multidimensional resolution.**

I. INTRODUCTION

Wavelet expansions have been the focus of many research papers. One of the reasons for their popularity in that they provide a more efficient representation of functions than other orthogonal expansions. Y. Meyer studied the convergence of orthogonal wavelet expansions. He showed that if the mother wavelet is r−regular, the orthogonal wavelet expansion of a function will converge to it in the sense of $L^p(R)$, $1 \leq p < \infty$, and in the sense of some Sobolev spaces.

A function $g(x)$, $x \in R^d$, $d \geq 1$, is said to be r−regular (in the sense of Meyer) if

$$|D^\alpha g(x)| \leq \frac{K_{\alpha,m}}{(1+|x|)^m}$$

for all α with $|\alpha| \leq r$ and $m = 0, 1, 2, \ldots$, where $K_{\alpha,m}$ are constants. Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index with $\alpha_i (i = 1, \ldots, d)$ being a non-negative integer and $|\alpha| = \sum_{i=1}^{d} \alpha_i$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots\ldots \partial x_d^{\alpha_d}}$$

S. Kelly, M. Kon and L. Raphael ([1],[2]) extended Walter's results by proving point wise convergence of orthogonal wavelet expansions in n dimensions. The key of their proofs is the following definition.

Definition 1.1 A bounded function $w : [0,\infty) \to R^+$ is a radial decreasing L^1 − ma- jorant of a given function h defined on R if $|h(x)| \leq w(|x|)$ and w satisfying the following conditions: (i) $w \in L^1([0,\infty))$, (ii) w is decreasing, (iii) $w(0) < \infty$.

The boundedness of w follows from (i) and (ii).

In all above work on point wise convergence it is essential that the summation kernel of the wavelet series given by

$$P_m(x,y) = \sum_k \varphi_{m,k}(x)\,\overline{\varphi_{m,k}(y)}$$

$$= 2^m \sum_{k\in Z} \varphi(2^m x - k)\,\overline{\varphi(2^m y - k)}$$

$$= 2^m K_\varphi(2^m x, 2^m y),$$

where $\varphi_{m,k}(x) = 2^{m/2}\varphi(2^m x - k)$ and $K_\varphi(x, y) = \sum_{k\in Z} \varphi(x-k)\,\overline{\varphi(y-k)}$, be absolutely bounded by redial decreasing L^1 – majorant dilation kernel i.e.

$\sum_{k\in Z} w(|x-k|)w(|y-k|) \leq c w\left(\frac{|x-y|}{2}\right)$ $x, y \in \mathbb{R}$, C is some constant depends on w. There are, however, some mother wavelets that does not satisfy these conditions. The summation kernel associated to scaling function $\varphi(t) = \frac{\sin \pi t}{\pi t}$ is seen to be

$$\frac{\sin \pi(t-y)}{\pi(t-y)} = \sum_{k=-\infty}^{\infty} \frac{\sin \pi(t-k)\sin \pi(y-k)}{\pi(t-k)\pi(y-k)}$$

It is clear that this kernel not belongs to $L^1(R)$. Hence this can not be absolutely bounded by radial decreasing L^1−majorant function.

The point wise convergence of the Shannon wavelet series can be studied directly but it is very special case and of less interest. In this paper we shall study the point wise conver- gence of prolate spheroidal wavelet expansions in $L^p(R)$, $1 \leq p < \infty$. Here the sinc function

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is replaced by one based on prolate spheroidal wave functions (PSWF's) which have much better time localization than the sinc function. The new wavelets preserve the high energy concentration in both the time and frequency domain inherited from PSWF's.

The connection between PSWF's and the Shannon

sampling theorem (Shannon [4]) given the formula

$$\sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}.$$

It holds for π−band limited signals with finite energy, that is, for continuous functions in $L^2(R)$ whose Fourier transform has support in $[-\pi,\pi]$. This theorem has become a well- known part of both the mathematical and engineering literature.

The sinc function is closely related to the PSWF's $\phi_{n,\sigma,\tau}(\tau)$. They constitute an orthonor- mal basis of the space of σ−band limited functions on the real line. They are concentrated on the interval $[-\tau, \tau]$ and depend on the two parameters σ and τ.

In order to construct PSWF wavelets, we begin with a scaling function φ whose integer translates are a Riesz basis of a space V_0. This space is usually taken to be a subspace of $L^2(R)$. Here we shall take $\varphi(x) = \phi_{0,\varphi,\tau}(x)$, where τ is any positive number. With this choice the space V_0 will turn out to be the Paley-Wiener space B_π of π−band limited functions no matter what the choice of τ.

This space then becomes part of a family of nested subspace usually referred as a multi-resolution analysis (MRA). The other spaces are obtained by dilations by factors of two: $f(t) \in V_m$, if and only if, $f(2^{-m} t) \in V_0$. These have the usual properties of an MRA

(i) $\ldots \subseteq V_{m-1} \subseteq V_m \subseteq \cdots \subseteq L^2(R)$,

(ii) $\bigcup V_m = L^2(R)$,

(iii) $\bigcap V_m = \{0\}$. The MRA consisting of the Paley-Wiener spaces ($V_m = B_{2^m\pi}$) has been widely studied and has its standard scaling function the sinc function $S(t) = \frac{\sin \pi t}{\pi t}$. This function has very good frequency localization, but not very good time localization. This has limited its use as a wavelet basis in comparison to the Daubechies wavelets which have compact support in the time domain. Because of the properties of entire functions, no band limited function has compact support in the time domain. However, the PSWF's are as close to it as one can get and in fact, for τ sufficiently large, can be made arbitrarily small outside of the interval of concentration. Hence they should be similar to the Daubichies wavelets for practical computation and superior to the sinc functions. Our basis will differ from the standard wavelet basis for V_0 consisting of translates of the sinc function.

The PSWF $\phi_{0,\pi,\tau}$ is a candidate for a scaling function with $V_0 = B_\pi$. There are several ways of constructing bases of the other subspaces $V_m = B_{2^m\pi}$ from those of V_0. One uses the standard wavelet approach in which dilations of $\phi_{0,\pi,\tau}$ i.e., $\phi_{0,\pi,\tau}(2^m t)$ are used to get the basis $\{\phi_{0,\pi,\tau}(2^m t - n)\}$ of V_m. In this case we get

It is also possible to find a dual Riesz basis for $\{\varphi_{0,\pi,\tau}(t-n)\}$.

We can get it by defining the Fourier transform of the dual function $\tilde{\varphi}_{0,\pi,\tau}(t)$ as

$$\tilde{\tilde{\varphi}}_{0,\pi,\tau}(w) = \frac{\varphi_{0,\pi,\tau}(w)}{\sum_k |\tilde{\varphi}_{0,\pi,\tau}(w-2\pi k)|^2}$$

$$\sum_k \tilde{\tilde{\varphi}}_{0,\pi,\tau}(w-2\pi k)\overline{\tilde{\varphi}_{0,\pi,\tau}(w-2\pi k)} = 1$$

It follows that $\{\tilde{\varphi}_{0,\pi,\tau}(t-n)\}$ is bi-orthogonal to $\{\varphi_{0,\pi,\tau}(t-n)\}$. Again, because $\tilde{\tilde{\varphi}}_{0,\pi,\tau}(w)$ is Positive on $[-\pi,\pi]$, it follows that $\{\tilde{\varphi}_{0,\pi,\tau}(t-n)\}$ is a Riesz basis of B_π.

The approximation of a function in $L^2(R)$ by function in V_m is given by a series of the

Form

$$f(x) = \sum_n \langle f, \tilde{\varphi}_{m,n} \rangle \varphi_{m,n}(x)$$

The kernel of this projection is given

$$q_m(x,t) = \sum_n \emptyset_{m,n}(2^m x - 2^{-m} n)\tilde{\emptyset}_{m,n}(2^m t - 2^{-m} n)$$

$$= 2^m \sum_n \varphi_{0,\pi,\tau}(2^m x - 2^{-m} n)\tilde{\varphi}_{0,\pi,\tau}(2^m t - 2^{-m} n)$$

$$= 2^m q_0(2^m x, 2^m t)$$

$$= \lambda q_0(\lambda x, \lambda t), \qquad \lambda = 2^m.$$

Let us consider the class $S(R)$ of rapidly decreasing C^∞− functions on R such that

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$$S(R) = \left\{f: R \to R, \sup_{x \epsilon R}\left(x^n \frac{d^m}{dx^m} f\right)(x) < \infty\right\}, n, m \in N \cup \{0\}$$

Then for f $\in$ S(R), the Fourier transform of f is given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_R e^{-i\xi x} f(x)\, dx.$$

It can be easily seen that of

$f \,\epsilon\, S(R) and \hat{f}(\xi) \epsilon\, S(R) and\, S(R)$ is dense in $L^p(R)$, $1 \le p < \infty$. Also Fourier transform is isometric in S(R).

The inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_R \hat{f}(\xi) e^{i\xi x}\, d\xi.$$

Let $[-2^m\pi, +2^m\pi]$ denote the support of $\hat{\Phi}$, is of the form

$$[-2^m\pi, 2^m\pi] = \cup_{i=1}^{n} [a_i, b_i] \cup O,$$

Where O is a set of measure zero. Then for f $\in V_0$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_E \hat{f}(\xi) e^{i\xi x}\, d\xi.$$

$$= \frac{1}{2\pi} \int_{-2^m\pi}^{2^m\pi} \int_R f(x) e^{-it\xi}\, dt e^{i\xi x}\, d\xi$$

$$= \int_R f(t) \left\{\frac{1}{2\pi} \int_{-2^m\pi}^{2^m\pi} e^{i\xi(x-t)}\, d\xi\right\} dt$$

$$= \int_R f(t)\, k(t,x) dt$$

Where

$$k(t,x) = \frac{1}{2\pi} \int_{-2^m\pi}^{2^m\pi} e^{i\xi(x-t)}\, d\xi = k(x-t).$$

The integral in (6.1.1) is absolutely convergent by the Cauchy-Schwarz inequality because

both f and k are in $L^2(R)$.

Definition 1.2. If $\varphi \epsilon L^1(R)$ $with$ $\hat{\varphi}(0) = 1$ and we define $\varphi_n(x) = n\varphi(nx) where\ n \to \infty$.

Then the sequences of functions $\{\varphi_n\}_{n=1}^{\infty}$ is an approximate identity if:

(1) $\int_R \varphi_n(x) dx = 1$ for all n.

(2) $\mathrm{Sup}_n \int_R |\varphi_n(x)| dx < \infty$,

(3) $\lim_{n\to\infty} \int_{|x|>\delta} |\varphi_n(x)|\, dx = 0$ $for\ every\ \delta > 0$.

Remark 1.1. If $00 \le \varphi(x) \epsilon\, S(R)$. Then $\varphi_n(x) = n\varphi(nx)$ is an approximate identity.

By the hypothesis the reproducing kernel series

$$\sum_n \emptyset_{m,n}(2^m x - 2^{-m} n) \tilde{\emptyset}_{m,n}(2^m t - 2^{-m} n)$$

Converges absolutely and uniformly for all x and t to a function $q_0(t, x)$. Thus, we have

$k(t,x) = \sum_n \varphi_{0,\pi,\tau}(x-n)\, \hat{\varphi}_{0,\pi,\tau}(t-n) = q_0(t,x)$ almost everywhere. Now we shall prove

Lemma 1.1 If $q_0(t,x) \in L^1(R)$with $\int_R q_0(t,x)dt = 1$. Then $q_m(t,x)$ is an approximate identity if

(1) $\int_R q_m(t,x)dt = 1$ for all m.

(2) $\mathrm{Sup}_m \int_R |q_m(t,x)| dt < \infty$,

(3) $\lim_{m\to\infty} \int_{|x-t|>\delta} |q_m(t,x)|\, dt = 0$ $for\ every\ \delta > 0$.

Proof 1. We see that $\int_R q_m(t,x)dt = \int_R \lambda q_0(\lambda(x-t))\, dt = \int_R q_0(\lambda(|x-t|)) d(\lambda|x-t|) = 1$.

Proof 2. We have $q_0(\xi) = \frac{1}{2\pi|\xi|} \ge \frac{1}{\sqrt{2\pi}} \left|\frac{e^{-i\xi b} - e^{-i\xi a}}{(-i\xi)}\right| = \left|\frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\xi x}\, dx\right|$,

$$\lambda q_0(\lambda|\xi|) \ge \sum_{i=1}^{m} \gamma_i \left|\frac{e^{-i\xi b} - e^{-i\xi a}}{\sqrt{2\pi}(-i\xi)}\right| = \sum_{i=1}^{m} \gamma_i \int_{\lambda a_i}^{\lambda b_i} \chi\Delta_i(x).$$

If $f \epsilon L^1(R)$ then $\log_{m\to\infty} \sum_{i=1}^{m} \gamma_i \chi\Delta_i(x)$ is in $L^1(R)$. Given $\epsilon > 0$, find $\sum_{i=1}^{m} \gamma_i \chi\Delta_i(x)$ such that

$$\int_R \left|f(x) - \sum_{i=1}^{m} \gamma_i \chi\Delta_i(x)\right| dx < \frac{\epsilon}{2}.$$

So we have

$$|\hat{f}(\xi)| \le \left|\frac{1}{\sqrt{2\pi}} \int_R e^{-i\xi x} f(x) dx\right|$$

$$\le \frac{1}{\sqrt{2\pi}} \int_R \left\{\left|f(x) - \sum_{i=1}^{m} \gamma_i \chi\Delta_i(x)\right| dx + \left|\sum_{i=1}^{m} \gamma_i \chi\Delta_i(x)\right|\right\} e^{-i\xi x} dx$$

$$\le \frac{\epsilon}{2\sqrt{2\pi}} + \frac{\sum_i \gamma_i}{|\xi|} < \frac{\epsilon}{2\sqrt{2\pi}}$$

Hence $\lambda q_0(\lambda|\xi|)$ is bounded for all ξ and λ. This proves (2).

Proof 3. We have

$$\int_{|x-t|>\delta} q_m(t,x) dt = \int_{|x-t|>\delta} \lambda q_0 \lambda(|x-t|)\, dt$$

$$= \int_{\delta}^{\infty} \lambda q_0(\lambda(|x-t|)) dt + \int_{-\infty}^{-\delta} \lambda\, q_0(\lambda|x-t|)\, dt$$

Substituting $x - y = \lambda(x-t)$

$$\lim_{m\to\infty} \int_{m\delta}^{\infty} q_0(x-y) dy + \int_{-\infty}^{-m\delta} q_0(x-y) dy = 0.$$

Remark 1.2. Let $0 \le q_0(t, x) \in S(R)$. Then $q_m(t, x) = \lambda\, q_0(\lambda|x-t|)$ is an approximate identity.

Lemma 1.2. If f $\in L^1(R)$ and $q_m(t, x) \in S(R)$ then

$q_m(t,x) * f \in S(R)$.

Proof. We have

$$(q_m(t,x)f) = \int_R q_m(x-t-y) f(y)dy$$
$$= \int_R q_m(y-t) f(x-y)dy$$

Or

$$\frac{d^n}{dx^n}(q_m(t,x)f) = \int_R q_m(y-t)\frac{d^n}{dx^n} f(x-y)dy$$
$$|x|^n \frac{d^n}{dx^n}(q_m(t,x)f)$$
$$= |x|^n \int_R f(x-y)\frac{d^n}{dx^n} q_m(y-t)dy,$$

Substituting $x-y=z$, it gives that

$$= \int_R f(y)|x|^n \frac{d^n}{dx^n} q_m(x-y-t)dy$$

Since $|x-y-t| \le |x-t| + |y| \le \frac{3|x-t|}{2}$, so from above we obtain

$$\int_{|y|>\frac{|x-t|}{2}} f(y)|x|^n \frac{d^n}{dx^n} q_m(x-y-t)dy + \int_{|y|<\frac{|x-t|}{2}} f(y)|x|^n \frac{d^n}{dx^n} q_m(x-y-t)dy \to 0.$$

Hence the proof is completed.

II. MAIN RESULTS

In this section we shall prove our main theorems.

Theorem 2.1 If $\{q_m(t,x)\}_{m=1}^{\infty}$ is an approximate identity then

$$\lim_{m\to\infty} \|f * q_m(t,x) - f\|_2 = 0 \text{ for every } f \in L^2(R)$$

Proof. Let us consider

$$\left[\int_R |(q_m(t,x)f)(x) - f(x)|^p dx\right]^{1/2}$$
$$= \left[\int_R dx \left|\int_R q_m(x-t-y)f(y)dy - f(x)\right|^2\right]^{1/2}$$
$$= \left[\int_R dx \left|\int_R q_m(y-t)f(x-y)dy - f(x)\right|^2\right]^{1/2}.$$

Since $f(x) = \int_R f(x)q_m(y-t)dy$; so we get

$$\left[\int_R dx \left|\int_R q_m(y-t)\big(f(x-y)-f(x)\big)dy\right|^2\right]^{1/2}$$
$$\le \left[\int_R dx \int_{|y-t|>\delta} |q_m(y-t)|^p\, |f(x-y)-f(x)|^2 dy\right]^{1/2}$$
$$+ \left[\int_R dx \int_{|y-t|\le\delta} |q_m(y-t)|^p\, |f(x-y)-f(x)|^2 dy\right]^{1/2}$$
$$\le \int_{|y-t|>\delta} dy\, |q_m(y-t)| \left[\int_R dx |f(x-y)-f(x)|^2 dy\right]^{1/2}$$
$$+ \int_{|y-t|\le\delta} dy |q_m(y-t)| \left[\int_R dx |f(x-y)-f(x)|^2 dy\right]^{1/2}$$
$$\le \int_{|y-t|>\delta} dy\, |q_m(y-t)|(2\|f\|_2)$$
$$+ \int_{|y-t|\le\delta} dy\, |q_m(y-t)| \sup_{|y|<\delta}\left[\int_R |f(x-y)-f(x)|\right]$$
$$\to 0 \text{ as } m\to\infty.$$

Hence the proof is completed.

Theorem 2.2. If $q_m(t,x)$ is an approximate identity and $f \in L^2(R)$ then the wavelet series

$$f_m(x) = \sum_k \langle f, \hat{\phi}_{m,n}\rangle\, \phi_{m,n}(x)$$

converges to f(x) as m → ∞ at every point of continuity of f(x).

Proof. The projection of $f \in L^2(R)$ on the space V_m is given by

$$f_m(x) = \int_R f(t)q_m(t,x)dt$$
$$= \int_R f(t)q_m(x-t)dt$$
$$= (f * q_m)(x)$$
$$\to f(x), (q_m \textit{ is an approximate identity}).$$

Also we have

$$f_m(x) = \int_R f(t) q_m(x-t)dt$$

$$= \sum_n \int_R f(t)\hat{\phi}_{m,n}(t)\phi_{m,n}(x)dt$$

$$= \sum_n \langle f, \hat{\phi}_{m,n}\rangle \phi_{m,n}(x)$$

Thus if $\mathbf{f} \in L^2(R)$, $q_m(x) \in S(R)$ then $q_m * f \in S(R)$ and S(R) is dense in $L^2(R)$

Then

$$\|f_m(x) - f(x)\|_{L^p(R)} \to 0 \; as \; m \to \infty.$$

This proves the theorem.

If the coefficients are given by the sampled values of the functions, then the convergence may not be so rapid, but has other nice properties. The approximation in V_m is now given by the hybrid series

$$f_m^s(t) = \sum_k f(2^{-m}k)\frac{\phi_m(t-2^{-m}k)}{2^m\hat{\phi}_m(0)}$$

Or we can write

$$f_m^s(t) = \int f(t)k_m(x,t)dt$$

Where

$$k_m(x,t) = \sum_k \frac{\phi_m(x-2^{-m}k)}{2^m\hat{\phi}_m(0)}\delta(t-2^{-m}k).$$

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