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Some Double Almost $(\lambda_m \mu_n)$ convergence in χ^2 −Riesz spaces defined by Musielak-Orlicz functions

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Abstract. **In this paper we introduce and study the notions of almost $(\lambda_m \mu_n)$ convergence in χ^2 −Riesz spaces; strongly $P-$ convergence , Cesaro strongly $P-$ convergence with respect to a Musielak-Orlicz functions and examine some properties of the resulting sequence spaces. We also introduce and study the statistical convergence of almost $(\lambda_m \mu_n)$ convergence in χ^2 − Riesz spaces and also some inclusion theorems are discussed.**

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I. INTRODUCTİON

We denote by w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We denote by w^2 the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.
Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1,2,3, \dots) \; .$$

A double sequence $x = (x_{mn})$ is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences is usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences is usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 be a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1,2,3, \dots \right\}, \qquad (1.1)$$

for all $x = \{x_{mn}\}$and$y = \{y_{mn}\}in$ Γ^2. Let ϕ denote the set of all finite sequences;
Consider a double sequence $x = (x_{mn})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \dots 0 & 0 & \dots \\ 0 & 0 & \dots 0 & 0 & \dots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \dots 1 & 0 & \dots \\ 0 & 0 & \dots 0 & 0 & \dots \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2.

II. DEFİNİTİONS AND PRELİMİNARİES

A double sequence $x = (x_{mn})$ has limit 0 (denoted by $P - limx = 0$)

(i.e) $\left((m + n)! |x_{mn}|\right)^{1/m+n} \to 0$ as $m, n \to \infty$. We shall write more briefly as $P - convergent \;\; to$ 0.

An Orlicz function is a function $M: [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function M is replaced by $M(x+y)\leq M(x)+M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy Δ_2- condition for all values u, if there exists $K>0$ such that $M(2u)\leq KM(u), u\geq 0$.

2.1 Lemma

Let M be an Orlicz functions which satisfies Δ_2- condition and let $0<\delta<1$. Then for each $t\geq\delta$, we have $M(t)<K\delta^{-1}M(2)$ for some constant $K>0$.
The concept of Orlicz sequence spaces was investigate by many authors, for some examples can be found at [31,32,33,34,35,36].
A double sequence $M=(M_{mn})$ of Orlicz function is called a Musielak-Orlicz function [see [29,30]]. A double sequence $g=(g_{mn})$ defined by

$$g_{mn}(v)=sup\{|v|u-(M_{mn})(u): u\geq 0\}, m,n=1,2,\cdots$$

is called the complementary functions of a sequence of Musielak-Orlicz M. For a given sequence of Musielak-Orlicz functions M, the Musielak-Orlicz sequence space t_M is defined as follows

$$t_M=\left\{x\in w^2: I_M(|x_{mn}|)^{1/m+n}\to 0 \quad as \quad m,n\to\infty\right\},$$

where I_M is a convex modular defined by

$$I_M(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}M_{mn}(|x_{mn}|)^{1/m+n}.$$

2.2 Definition

A double sequence $x=(x_{mn})$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$P-lim_{p,q\to\infty}sup_{r,s\geq 0}\frac{1}{pq}\sum_{m=r}^{r+p-1}\sum_{n=s}^{s+q-1}\left((m+n)!\,|x_{mn}|\right)^{1/m+n}=0.$$

that is, the average value of (x_{mn}) taken over any rectangle $\{(m,n): r\leq m\leq r+p-1, s\leq n\leq s+q-1\}$ tends to 0 as both p and q to ∞, and this $P-$ convergence is uniform in r and s. Let denote the set of sequences with this property as $\left[\widehat{\chi^2}\right]$.

2.3 Definition

Let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and

$$\lambda_{m+1}\leq\lambda_m+1, \lambda_1=1, \quad \mu_{n+1}\leq\mu_n+1, \mu_1=1.$$

Let $I_m=[m-\lambda_m+1,m]$ and $I_n=[n-\mu_n+1,n]$.

For any set $K\subseteq\mathbb{N}\times\mathbb{N}$, the number

$$\delta_{\lambda,\mu}(K)=lim_{m,n\to\infty}\frac{1}{\lambda_m\mu_n}|\{(i,j): i\in I_m, j\in I_n, (i,j)\in K\}|,$$

is called the $(\lambda,\mu)-$ density of the set K provided the limit exists.

2.4 Definition

A double sequence $x=(x_{mn})$ of numbers is said to be $(\lambda,\mu)-$ statistical convergent to a number ξ provided that for each $\varepsilon>0$,

$$lim_{m,n\to\infty}\frac{1}{\lambda_m\mu_n}|\{(i,j): i\in I_m, j\in I_n, |x_{mn}-\xi|\geq\varepsilon\}|=0,$$

that is, the set $K(\varepsilon)=\frac{1}{\lambda_m\mu_n}|\{(i,j): i\in I_m, j\in I_n, |x_{mn}-\xi|\geq\varepsilon\}|$ has $(\lambda,\mu)-$ density zero. In this case the number ξ is called the $(\lambda,\mu)-$ statistical limit of the sequence $x=(x_{mn})$ and we write $St_{(\lambda,\mu)}lim_{m,n\to\infty}x_{mn}=\xi$.

2.5 Definition

Let M be an Orlicz function and $P=(p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space:

$$\chi^2_M\left[AC_{\lambda\mu},P\right]=\left\{P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left((m+n)!\,|x_{m+r,n+s}|\right)^{1/m+n}\right]^{p_{mn}}=0,\right\},$$ uniformly in r and s.

We shall denote $\chi^2_M\left[AC_{\lambda_m\mu_n},P\right]$ as $\chi^2\left[AC_{\lambda_m\mu_n}\right]$ respectively when $p_{mn}=1$ for all m and n. If x is in $\chi^2\left[AC_{\lambda_m\mu_n},P\right]$, we shall say that x is almost $(\lambda_m\mu_n)$ in χ^2 strongly $P-$convergent with respect to the Orlicz function M. Also note if $M(x)=x, p_{mn}=1$ for all m,n and k then $\chi^2_M\left[AC_{\lambda_m\mu_n},P\right]=\chi^2\left[AC_{\lambda_m\mu_n},P\right]$, which are defined as follows:

$$\chi^2\left[AC_{\lambda_m\mu_n},P\right]=\left\{P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left((m+n)!\,|x_{m+r,n+s}|\right)^{1/m+n}\right]=0,\right\},$$ uniformly in r and s.

Again note if $p_{mn}=1$ for all m and n then $\chi^2_M\left[AC_{\lambda_m\mu_n},P\right]=\chi^2_M\left[AC_{\lambda_m\mu_n}\right]$. We define

$$\chi^2_M\left[AC_{\lambda_m\mu_n},P\right]=\left\{P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left((m+\right.\right.\right.$$

$n)!\,|x_{m+r,n+s}|)^{1/m+n}\Big]^{p_{mn}} = 0,\Big\}$, uniformly in r and s.

2.6 Definition

Let M be an Orlicz function and $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space: $\chi^2_M[P] = \Big\{P - lim_{p,q\to\infty} \frac{1}{pq}\sum_{m=1}^{p}\sum_{n=1}^{q}\Big[M\Big((m+n)!\,|x_{m+r,n+s}|\Big)^{1/m+n}\Big]^{p_{mn}} = 0\Big\}$, uniformly in r and s.

If we take $M(x) = x, p_{mn} = 1$ for all m and n then $\chi^2_M[P] = \chi^2$.

2.7 Definition

The double number sequence x is $\widehat{S_{\lambda_m\mu_n}} - P -$ convergent to 0 then

$$P - lim_{m,n}\frac{1}{\lambda_m\mu_n} max_{r,s}\left|\left\{(m,n) \in I_{r,s}: M\left((m+n)!\,|x_{m+r,n+s} - 0|\right)^{1/m+n}\right\}\right| = 0.$$

In this case we write $\widehat{S_{\lambda_m\mu_n}} - lim\big(M(m+n)!\,|x_{m+r,n+s} - 0|\big)^{1/m+n} = 0.$

III. THE DOUBLE ALMOST $(\lambda_m\mu_n)$ İN χ^2 İN RİESZ SPACE

Let $n \in \mathbb{N}$ and d_p on X, a real vector space of dimension m and d_p applied to $(x_1,\cdots,x_n)$ with $n \le m$. A real valued function $d_p(x_1,\dots,x_n) = \|(d_1(x_1,0),\dots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_p(x_1,0),\dots,d_p(x_n,0))\|_p = 0$ if and and only if $d_p(x_1,0),\dots,d_p(x_n,0)$ are linearly dependent,

(ii) $\|(d_p(x_1,0),\dots,d_p(x_n,0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_p(x_1,0),\dots,\alpha d_p(x_n,0))\|_p = |\alpha|\,\|(d_p(x_1,0),\dots,d_p(x_n,0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)) = \left(d_p(x_1,x_2,\cdots x_n)^p + d_p(y_1,y_2,\cdots y_n)^p\right)^{1/p} for 1 \le p < \infty$; (or)

(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)) := \sup\{d_p(x_1,x_2,\cdots x_n), d_p(y_1,y_2,\cdots y_n)\}$,

for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_p(x_1,0),\dots,d_p(x_n,0))\|_E = sup\left(|det(d_p(x_{mn},0))|\right) = sup\left(\begin{vmatrix} d_p(x_{11},0) & d_p(x_{12},0) & \dots & d_p(x_{1n},0) \\ d_p(x_{21},0) & d_p(x_{22},0) & \dots & d_p(x_{1n},0) \\ . & & & \\ . & & & \\ . & & & \\ d_p(x_{n1},0) & d_p(x_{n2},0) & \dots & d_p(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

3.1 Definition

Let L be a real vector space and let $\le$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties :

(i) If $x, y \in L$ and $y \le x$, then $y + z \le x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \le x$, then $\lambda y \le \lambda x$ for each $\lambda \ge 0$.

If in addition L is a lattice with respect to the partial ordering, then L is said to be Riesz space.

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \le |y|$ implies $x \in S$.

A linear topology τ on a Riesz space X is said to be locally solid if τ has a base at zero consisting of solid sets.

3.2 Definition

Let $\chi^{2\tau}_M\left[AC_{\lambda\mu}, P, \left\|\left(d_p(x_1,0), d_p(x_2,0),\cdots,d_p(x_{n-1},0)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz functions. A sequence (x_{mn}) of points in χ^2 is said to be $S(\tau)-$ convergent to an element 0 of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n \in \mathbb{N}: M_{mn}\left(\left((m+n)!\,|x_{mn}|\right)^{1/m+n}\right) \notin V\right\}\right) = 0$$

that is , $\Big(P - lim_{m,n,r,s}\frac{1}{\lambda_m\mu_n}\Big\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\Big[M\Big((m+$

$n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\Big]^{p_{mn}} \notin V\Big\}\Big) = 0,$ uniformly with respect to r and s.

In this case we write

$$S(\tau) - \Big(P - lim_{m,n,r,s} \frac{1}{\lambda_m \mu_n} \Big\{ \sum_{m\in I_{r,s}} \sum_{n\in I_{r,s}} \Big[M\Big((m+n)!\left|x_{m+r,n+s}\right|\Big)^{1/m+n}\Big]^{p_{mn}} \notin V\Big\}\Big) = 0,$$

uniformly with respect to r and s.

3.3 Definition

Let $\chi_M^{2\tau}\left[AC_{\lambda\mu}, P, \left\|\left(d_p(x_1,0), d_p(x_2,0), \cdots, d_p(x_{n-1},0)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz functions. A sequence (x_{mn}) of points in χ^2 is said to be $(\lambda_m \mu_n)-$ sequence convergent to an element 0 of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n \in \mathbb{N}: M_{mn}\left(\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) \notin V\right\}\right) = 0$$

that is,

$$\Big(P - lim_{m,n,r,s} \frac{1}{\lambda_m \mu_n} \frac{1}{v_r v_s} \Big\{ \sum_{m\in I_{r,s}} \sum_{n\in I_{r,s}} \Big[M\Big((m+n)!\left|x_{m+r,n+s}\right|\Big)^{1/m+n}\Big]^{p_{mn}} \notin V\Big\}\Big) = 0.$$

In this case we write

$$AL_{\lambda\mu}(\tau) - \Big(P - lim_{m,n,r,s} \frac{1}{\lambda_m \mu_n} \frac{1}{v_r v_s} \Big\{ \sum_{m\in I_{r,s}} \sum_{n\in I_{r,s}} \Big[M\Big((m+n)!\left|x_{m+r,n+s}\right|\Big)^{1/m+n}\Big]^{p_{mn}} \notin V\Big\}\Big) = 0.$$

Where $AL_{\lambda\mu}(\tau)$ represents is a double almost $(\lambda\mu)$ strongly summable in χ^2 −Riesz space defined by a Musielak-Orlicz functions.

3.4 Definition

Let $\chi_M^{2\tau}\left[AC_{\lambda\mu}, P, \left\|\left(d_p(x_1,0), d_p(x_2,0), \cdots, d_p(x_{n-1},0)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz functions. A sequence (x_{mn}) of points in χ^2 is said to be Cesa'ro strongly summable almost $(\lambda\mu)-$ sequence convergent to an element 0 of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n \in \mathbb{N}: M_{mn}\left(\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) \notin V\right\}\right) = 0$$

that is,

$$\Big(P - lim_{m,n,r,s} \frac{1}{\lambda_m \mu_n} \frac{1}{pq} \Big\{ \sum_{m}^{p} \sum_{n}^{q} \Big[M\Big((m+n)!\left|x_{m+r,n+s}\right|\Big)^{1/m+n}\Big]^{p_{mn}} \notin V\Big\}\Big) = 0.$$

In this case we write

$$\sigma_{\lambda_m \mu_n}(\tau) - \Big(P - lim_{p,q,m,n,r,s} \frac{1}{\lambda_m \mu_n} \frac{1}{pq} \Big\{ \sum_{m=1}^{p} \sum_{n=1}^{q} \Big[M\Big((m+n)!\left|x_{m+r,n+s}\right|\Big)^{1/m+n}\Big]^{p_{mn}} \notin V\Big\}\Big) = 0.$$

Where $\sigma_{\lambda\mu}(\tau)$ represents is a double Cesa'ro strongly summable almost $(\lambda\mu)$ sequence of χ^2- Riesz space defined by a Musielak-Orlicz functions.

3.5 Definition

Let $\chi_M^{2\tau}\left[AC_{\lambda\mu}, P, \left\|\left(d_p(x_1,0), d_p(x_2,0), \cdots, d_p(x_{n-1},0)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz functions. A sequence (x_{mn}) of points in χ^2 is said to be $S(\tau)-$ statistically convergent to an element 0 of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n \in \mathbb{N}: M_{mn}\left(\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) \notin V\right\}\right) = 0$$

that is,

$$\Big(P - lim_{p,q,m,n,r,s} \frac{1}{\lambda_m \mu_n} \frac{1}{pq} \quad card\Big\{(m,n): m \le p, n \le q \ \ and \ \ \Big[M\Big((m+n)!\left|x_{m+r,n+s}\right|\Big)^{1/m+n}\Big]^{p_{mn}} \ge \varepsilon \notin V, \ for \ \ each \ \ \varepsilon > 0\Big\}\Big) = 0.$$

In this case we write

$$S(\tau)-\left(P-lim_{p,q,m,n,r,s}\frac{1}{\lambda_m\mu_n}\frac{1}{pq}\quad card\left\{(m,n):m\leq p,n\leq q\quad and\quad \left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\geq\varepsilon\notin V,\quad for\quad each\quad \varepsilon>0\right\}\right)=0.$$

3.6 Definition

Let $\chi_M^{2\tau}\left[AC_{\lambda\mu},P,\left\|\left(d_p(x_1,0),d_p(x_2,0),\cdots,d_p(x_{n-1},0)\right)\right\|_p\right]$ be a Riesz space of Musielak-Orlicz functions. A sequence (x_{mn}) of points in χ^2 is said to be $S_{\lambda m\mu_n}(\tau)-$ almost $(\lambda\mu)$ statistically convergent to an element 0 of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n\in\mathbb{N}:M_{mn}\left(\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right)\notin V\right\}\right)=0$$

that is, $\left(P-lim_{m,n,r,s}\frac{1}{\lambda_m\mu_n}\frac{1}{v_rv_s}\quad card\left\{(m,n)\in I_{rs}\quad and\quad \left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\geq\varepsilon\notin V,\quad for\quad each\quad \varepsilon>0\right\}\right)=0.$

In this case we write

$$S_{\lambda_m\mu_n}(\tau)-\left(P-lim_{m,n,r,s}\frac{1}{\lambda_m\mu_n}\frac{1}{v_rv_s}\quad card\left\{(m,n)\in I_{rs}\quad and\quad \left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\geq\varepsilon\notin V,\quad for\quad each\quad \varepsilon>0\right\}\right)=0.$$

A double sequence $\lambda_m\mu_n=(\alpha_r,\beta_s)$ is said to be double almost if there exists sequence (α_r) and (β_s) of non-negative integers such that

$v_r=\alpha_r-\alpha_{r-1}\to\infty$ as $r\to\infty,\alpha_0=0$
$v_s=\beta_s-\beta_{s-1}\to\infty$ as $s\to\infty,\beta_0=0$

Let $v_{rs}=v_rv_s,\lambda_m\mu_n$ is obtain by $I_{rs}=\{(x,y):\alpha_{r-1}<x\leq\alpha_r\quad and\quad \beta_{s-1}<y\leq\beta_s\}$

IV. MAİN RESULTS

4.1 Theorem

Let $(\lambda_m\mu_n)$ be a double almost Riesz sequence space and M be a Musielak-Orlicz functions. Then if double almost $(\lambda_m\mu_n)$ convergence in χ^2 is strongly summable then it is almost $(\lambda_m\mu_n)$ statistically convergent.

Proof: Suppose χ^2 is double almost $(\lambda_m\mu_n)$ convergence in Riesz space of Musielak-Orlicz functions is strongly summable. Then,

$\left(lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\notin V\right\}\right)=0.$ Now the results follows from the following inequality

$$\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\geq\varepsilon card\left\{(m,n)\in I_{rs}:\left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\geq\varepsilon\right\}.$$

Remark: The converse of the above theorem is not true. For it, we consider the following example.

Let $\lambda_m\mu_n=(2^m,2^n);m,n=1,2,3,\cdots$. Then $S(\tau)=\sigma_{\lambda_m\mu_n}(\tau)$. Consider the sequence $x=(x_{mn})$ as follows:

If $s=i^2,i\in\mathbb{N}$ then,

$$\left((m+n)!\,x_{m+r,n+s}(t)\right)=\begin{cases}\left[1+\frac{t}{\sqrt{s}}\right]^{1/m+n}, & if\quad -\sqrt{s}\leq t\leq 0\\ \left[1-\frac{t}{\sqrt{s}}\right]^{1/m+n}, & if\quad 0\leq t\leq\sqrt{s}\\ 0, & if\quad otherwise\end{cases}$$

If $s\neq i^2$ then, $\left((m+n)!\,x_{m+r,n+s}(t)\right)=0.$

The sequence $x=(x_{mn})$ is almost $(\lambda_m\mu_n)$ statistically convergent to 0. But

$$\left(lim_{p,q,m,n,r,s}\frac{1}{\lambda_m\mu_n}\frac{1}{pq}\left\{\sum_{m=1}^{p}\sum_{n=1}^{q}\left[M\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right\}\right)\to\quad 1\quad as\quad p,q\to\infty.$$

4.2 Theorem

Let $(\lambda_m\mu_n)$ be a double almost Riesz sequence space and M be a Musielak-Orlicz functions. Then if double almost $(\lambda_m\mu_n)$ convergence in χ^2 is statistically convergent, then it is almost $(\lambda_m\mu_n)$ Cesa'ro strongly.

Proof: Suppose $x=(x_{mn})$ is bounded and statistically convergent to 0, we can find a number N

such that $\left[M\left(\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right] \leq \frac{N}{(m+n)!^{1/m+n}}$ for all $m,n \in \mathbb{N}$.

Since $x = (x_{mn})$ is statistically convergent to 0, for each $\varepsilon > 0$ such that

$$\left(lim_{p,q,m,n,r,s} \frac{1}{\lambda_m \mu_n} \frac{1}{pq} \quad card\left\{(m,n): m \leq p, n \leq q \quad and \quad \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} \geq \varepsilon\right\}\right) = 0.$$

Therefore

$$\left(\frac{1}{\lambda_m \mu_n} \frac{1}{pq}\left\{\sum_{1 \leq m \leq p} \sum_{1 \leq n \leq q} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right\}\right) \leq \left(\frac{1}{\lambda_m \mu_n} \frac{N}{pq} \quad card\left\{(m,n): m \leq p, n \leq q \quad and \quad \left[M\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} \geq \varepsilon\right\}\right) + \varepsilon.$$

4.3 Theorem

Let $(\lambda_m \mu_n)$ be a double almost Riesz sequence space and M be a Musielak-Orlicz functions. Then if double almost $(\lambda_m \mu_n)$ convergence in χ^2 is bounded and statistically convergent, then it is almost $(\lambda_m \mu_n)$ strongly.

Proof: Proof follows by similar arguments as applied to prove above theorem.

4.4 Theorem

Let $(\lambda_m \mu_n)$ be a double almost Riesz sequence space and M be a Musielak-Orlicz functions. Then if double almost $(\lambda_m \mu_n)$ convergence in χ^2 is bounded then it is almost $(\lambda_m \mu_n)$ statistically convergent if and only if strongly.

Proof: Proof follows by combining Theorem 4.1 and Theorem 4.3.

4.5 Theorem

$AL_{\lambda\mu}(\tau)$ is a complete metric under the metric g is defined by

$$g(x,0) = sup_{r,s}\left(\frac{1}{\lambda_m \mu_n} \frac{1}{v_r v_s}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|, 0\right)^{1/m+n}\right]^{p_{mn}}\right\}\right)$$

Proof: It is easy to see that $AL_{\lambda\mu}(\tau)$ is a metric.

To Prove completeness, let x^i be a Cauchy sequence in $AL_{\lambda\mu}(\tau)$, where $x^i = (x^i_{mn})$ for each $i \in \mathbb{N}$. Therefore for each $\varepsilon > 0$, there exist a positive integer n_0 such that

$$g(x^i,0) = sup_{r,s}\left(\frac{1}{\lambda_m \mu_n} \frac{1}{v_r v_s}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|, 0\right)^{1/m+n}\right]^{p_{mn}}\right\}\right) < \varepsilon \text{ for all } i \geq n_0.$$

It follows that

$$\frac{1}{\lambda_m \mu_n} \frac{1}{v_r v_s}\left\{\sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|, 0\right)^{1/m+n}\right]^{p_{mn}}\right\} < \varepsilon \text{ for all } i \geq n_0, \text{ for all } r,s \in \mathbb{N}.$$

Hence

$$\left[M\left((m+n)! \left|x_{m+r,n+s}\right|, 0\right)^{1/m+n}\right]^{p_{mn}} < \varepsilon \quad \text{for all } m,n \in \mathbb{N}.$$

This implies that (x^i_{mn}) is a Cauchy sequence in $L(R)$. But $L(R)$ is complete and so (x^i_{mn}) is convergent in $L(R)$.

Therefore let $lim_{i\to\infty} x^i_{mn} = x_{mn}$, exists for all $m,n \geq 1$.

This implies that $g(x^i,0) < \varepsilon$ for all $i \geq n_0$. This show that $x = (x_{mn}) \in AL_{\lambda_m \mu_n}(\tau)$.

4.6 Theorem

$\sigma_{\lambda\mu}(\tau)$ is a complete metric space under the metric g^* defined by

$$g^*(x,0) = sup_{r,s}\left(\frac{1}{\lambda_m \mu_n} \frac{1}{rs}\left\{\sum_{1 \leq m \leq r} \sum_{1 \leq n \leq s} \left[M\left((m+n)! \left|x_{m+r,n+s}\right|, 0\right)^{1/m+n}\right]^{p_{mn}}\right\}\right)$$

Proof: The proof of theorem is followed by theorem 4.5. In fact $\lambda_m \mu_n = (2^r, 2^s); r,s = 1,2,3,\cdots$, and $AL_{\lambda\mu}(\tau) = \sigma_{\lambda\mu}(\tau)$.

REFERENCES

[1]. T.Apostol, Mathematical Analysis, *Addison-wesley* , *London*, **1**978.

[2]. .J.I'A.Bromwich, An introduction to the theory of infinite series *Macmillan and Co.Ltd.* ,New York, **(**1965).

[3]. G.H.Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.

[4]. .J.Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc*, **100(1)** (1986),

161-166.
[5]. F.Moricz and B.E.Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.
[6]. M.Mursaleen,M.A.Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Math.* , **Vol. XXXII** (1999), 145-150.
[7]. H.Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1953), 29-49.
[8]. W.H.Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.
[9]. B.C.Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.
[10]. P.K.Kamthan and M.Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York* , **1**981.
[11]. M.Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.
[12]. M.Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 523-531.
[13]. Mursaleen and O.H.H. Edely,Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 532-540.
[14]. N.Subramanian and U.K.Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **46**, (2010).
[15]. B.Kuttner, Note on strong summability, *J. London Math. Soc.*, **21**(1946), 118-122.
[16]. I.J.Maddox, On strong almost convergence, *Math. Proc. Cambridge Philos. Soc.*, **85(2)**, (1979), 345-350.
[17]. J.Cannor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.*, **32(2)**, (1989), 194-198.
[18]. A.Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, *Math. Ann.*, **53**, (1900), 289-321.
[19]. H.J.Hamilton, Transformations of multiple sequences, *Duke Math. J.*, **2**, (1936), 29-60.
[20]. A Generalization of multiple sequences transformation, *Duke Math. J.*, **4**, (1938), 343-358.
[21]. Change of Dimension in sequence transformation , *Duke Math. J.*, **4**, (1938), 341-342.
[22]. Preservation of partial Limits in Multiple sequence transformations, *Duke Math. J.*, **4**, (1939), 293-297.
[23]. G.M.Robison, Divergent double sequences and series, *Amer. Math. Soc. Trans.*, **28**, (1926), 50-73.
[24]. B.Altay and F.BaSar, The fine spectrum and the matrix domain of the difference operator Δ on the sequence space $\ell_p, (0 < p < 1)$, *Commun. Math. Anal.*, **2(2)**, (2007), 1-11.
[25]. R.Colak,M.Et and E.Malkowsky, Some Topics of Sequence Spaces, Lecture Notes in Mathematics, *Firat Univ. Elazig, Turkey*, **2004**, pp. 1-63, Firat Univ. Press, (2004), ISBN: 975-394-0386-6.
[26]. E.Savas and Richard F. Patterson, On some double almost lacunary sequence spaces defined by Orlicz functions, *Filomat (Niš)*, **19**, (2005), 35-44.
[27]. N.Subramanian and A. Esi, The generalized triple difference of χ^3 sequence spaces , *Global Journal of Mathematical Analysis*, **3(2)**, (2015), 54-60.
[28]. N.Subramanian and A. Esi, The $\int \chi^{3\lambda I}$ statistical convergence of pre-Cauchy over the $p-$ metric space, *Annals of the University of Craiova - Mathematics and Computer Science Series*, **communicated**.
[29]. L.Maligranda, Orlicz spaces and interpolation, In:Seminars in Mathematics, 5, *Polish Academy of Sciences*, **1989**.
[30]. J. Musielak, Orlicz Spaces,Lectures Notes in Math.,1034, *Springer-Verlag*, **1983**.
[31]. A.Esi, M.I s k.,and A.Esi., “ On Some New Sequence Spaces Defined By Orlicz Functions” Indian J.Pure Appl.Math. **35 (1)**(2004),31-36.
[32]. A. Esi and M.N.Çatalbas, Some new generalized difference double sequence spaces via Orlicz functions, Scientia Magna, **7(3)**(2011), 59-68.
[33]. A. Esi, M.Acikgoz and Ayten Esi, On a class of generalized sequences related to the lp space defined by Orlicz function, Bol.Soc.Paran Mat. **31(1)**(2013), 113-123.
[34]. A. Esi,M.K.Özdemir, A.Esi, On some real valued I-convergent λ -summable difference sequence spaces defined by sequences of Orlicz functions, Inf.Sci.Lett.5(2)(2016), 47-51.
[35]. A.Esi and M.N.Çatalbas, Some sequence spaces of interval numbers defined by Orlicz functions, Proceedings of the Jangjeon Mathematical Society, **20(1)**(2017), 35-41.
[36]. A. Esi, B.Hazarika and A. Esi, New type of lacunary Orlicz difference sequence spaces generated by Infinite matricas, Filomat **30(12)** (2016), 3195-3208.