Integral Transform of Fractional Derivatives

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***Abstract*: In this paper we deal with various integral transform of Fractional derivative based on the Riemann-Liouville Derivatives, Riemann-Liouville Derivatives and Caputo's Fractional derivatives.We anticipate perspective applications of fractional Transform in physical systems.**

***Keyword:* Fractional Derivatives, Fourier Transform, Laplace Transform, Convolution of functions.**

I. INTRODUCTION :

The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions. The Fourier Transform therefore gives us a unique way of viewing any function - as the sum of simple sinusoids. its widespread popularity is due to its practical application in virtually every field of science and engineering. The Laplace transform is very similar[3] to the Fourier transform. While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Laplace transforms are usually restricted to functions of t with t > 0. A consequence of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Also techniques of complex variables can be used directly to study Laplace transforms. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function.

II. OVERVIEW: FRACTIONAL CALCULUS

Fractional Calculus is a term used for the theory of derivatives and integrals of arbitrary order, which generalize the notion of integer order differentiation and n-fold integration. The idea behind Fractional calculus is to generalize the definition of differentiation and integration with order $n \in \mathbb{N}$ to order $s \in \mathbb{R}$. The first discussion[9] on Fractional Calculus began in 1695 in a letter to L'Hopital by Leibniz in which he discussed about calculus of arbitrary order. Fractional Calculus is three centuries old. Few names that laid the foundation of Fractional Calculus are Abel, Liouville, Riemann, Euler, Caputo etc. Fractional Calculus has recently been applied in various areas of engineering, science, finance, applied mathematics and bio engineering.[10] . It has earlier been observed that derivatives of non-integer order are useful for describing the properties of various real materials like polymer, rocks etc. Also the fractional order models were found more logical to talk an discuss about than the integer-order models. In this paper we are focusing on Fractional Derivatives. Different people gave different definitions for the Fractional Derivative. Few definitions are :

Grunwald-Letnikov Fractional Derivatives: Let us consider a continous function f(t),We define

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) $$

The above formula has been obtained under thc assumption that the derivatives $f^{(k)}(t)$ **(k=1,2,3,**. . . ,m+1) are continuous in the closed interval [a,t] and that ***m is the integer number satisfying m>p-1***

Riemann-Liouville Derivatives:

$$ {}_aD_t^p f(t) = (\tfrac{d}{dt})^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \ (m \le p < m+1) $$

Caputo's Fractional Derivatives: The definition of the fractional differentiation of the Riemann-Liouville Derivatives type played an important role in the development of the theory

of fractional derivatives and for its applications in pure mathematics. However, the demands of modern technology require a certain revision of well established mathematical approach .The Caputo approach provides an interpolation between an integer order derivatives:

$$ {}^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(\alpha-n)} \int_a^x \frac{f^{(n)}(u)}{(x-u)^{(\alpha-n+1)}}, $$

$n-1 < \propto < n, \propto \in \mathbb{R}, n \in \mathbb{N}$

Euler's Fractional Derivatives:

$$ \frac{d^{\propto}}{dt^{\propto}}[t^{\beta}] = D_t^{\alpha}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \alpha \in \mathbb{R} $$

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Sequential Fractional Derivatives: The main idea of differentiation and integration of arbitrary order is the generalization of iterated integration and differentiation. In all these approaches we replace the integer valued parameter n of a operator denoted by $\frac{d^n}{dt^n}$ with a non integer perameter p.

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However ,we can assume that the n-th order differentiation is simply a series of n first order differentiation .So, considering more general expressions

$$D_t^{\alpha} = D_t^{\alpha_1} D_t^{\alpha_2} D_t^{\alpha_3} \dots\dots\dots . D_t^{\alpha_n}$$

Where $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \cdots \dots\dots . \alpha_n$
Which we will also call the sequential fractional derivatives.

Indeed, Riemann-Liouville Derivatives can be written as

$_aD_t^p f(t) = \frac{d}{dt}\frac{d}{dt} \dots\dots . \frac{d}{dt}\, {}_aD_t^{-(n-p)} f(t)$ (n-1$\leq p < n$)
While the Caputo fractional differential operator can be written as
${}^CD^{\alpha} f(x) = {}_aD_t^{-(n-p)} \frac{d}{dt} \dots\dots . \frac{d}{dt}$ f(t)(n-1$< p \leq n-1$)

Properties of Fractional Derivatives:
Fractional Derivatives satisfy almost all the properties that hold for[5] ordinary derivatives. We are aware of the general properties of the derivative operator $D_t^n, n \in \mathbb{N}$. Below mentioned are the properties of Fractional Derivative that can be easily verified:

- $D_t^{\propto}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[g(t)]$

where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.

- $D_t^{\propto}[f(t)C] = \sum_{k=0}^{\infty} \binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[C] = D_t^{\propto}[f(t)]C$.
- $D_t^{\propto}[h(t)+g(t)] = \sum_{k=0}^{\infty} \binom{\propto}{k} D_t^{\propto-k}[t^0] D_t^k[h(t)+g(t)] = D_t^{\propto}[h(t)] + D_t^{\propto}[g(t)]$.
- $D_t^{\propto}[h(at)] = a^{\propto} D_x^{\propto}[h(x)]\,, x = at$.
- $D_t^{\propto}[t^{-m}] = (-1)^{\propto} \frac{\Gamma(m+\propto)}{\Gamma(m)} t^{-(m+\propto)}$.
- $D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}(f(t))] = D_t^{\nu}[D_t^{\mu}(f(t))]$.

$D_t^{-1}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$, where $\alpha \in D_t^{\propto}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[g(t)]$, where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.

- $D_t^{\propto}[f(t)C] = \sum_{k=0}^{\infty} \binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[C] = D_t^{\propto}[f(t)]C$ where C is an arbitrary constant.
- $D_t^{\propto}[h(t)+g(t)] = \sum_{k=0}^{\infty} \binom{\propto}{k} D_t^{\propto-k}[t^0] D_t^k[h(t)+g(t)] = D_t^{\propto}[h(t)] + D_t^{\propto}[g(t)]$.
- $D_t^{\propto}[h(at)] = a^{\propto} D_x^{\propto}[h(x)]$ under the scaling $x = at$.
- $D_t^{\propto}[t^{-m}] = (-1)^{\propto} \frac{\Gamma(m+\propto)}{\Gamma(m)} t^{-(m+\propto)}$ for a given $m \in \mathbb{R}$.
- $D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}(f(t))] = D_t^{\nu}[D_t^{\mu}(f(t))]$ under the composition of D_t^{ν} and D_t^{μ} on $f(t)$.
- $D_t^{-1}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$, where $\beta \in \mathbb{R}$ corresponding to a negative order derivative.

III. INTEGRAL TRANSFORM OF FRACTIONAL DERIVATIVES:

In this section, we formulate Laplace and Fourier Transform of Fractional derivative of different approach discussed in above section.

3.1 Laplace Transforms OF Fractional Derivatives:
The laplace transform of a function f(t) is defined as
F(s) = L(f(t))=$\int_0^{\infty} e^{-st}$f(t) dt
For existence of Laplace transform of f(t), f(t) must be of exponential order.
the original f(t) can be obtained from F(s) with the help of [3]Inverse Laplace Transform
f(t) = $L^{-1}[F(s), t] = \int_{c-i\infty}^{c+i\infty} e^s F(s) ds$, c= Re(s)> c_0
Where c_0 lies in right half plane of the absolute convergence of Laplace integral.

Laplace Transform of convolution is defined as

f(t)*g(t) =$\int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t g(t-\tau) f(\tau) d\tau$

Another useful property which we are needed is Laplace Transform of derivative of an integer order n of a function f(t) :
L{$f^n(t)$;s}=$s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} \{f^n(0)$ = $s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0)$

Likewise, we can easily prove Laplace Transform of Fractional derivatives of order p> 0 in terms of Riemann-Liouville Derivatives
p
L{${}_0D_t^p f(t); s$}=s^pF(s)$- \sum_{k=0}^{n-1} s^k\, [{}_0D_t^{p-k-1} f(t)]$
(n-1$\leq p < n$)

Similarly, we can easily establish Laplace Transform Caputo Derivative as
L{${}^CD^{\alpha} f(x); s\} = s^p$F(s)$- \sum_{k=0}^{n-1} s^{p-k-1} \{f^n(0)\}$
(n-1$< p \leq n$)

3.2 Fourier Transform of Fractional Derivatives:
The exponential Fourier Transform of a continous function h(t) which is absolutely integrable in (-∞,∞) is given by

$$F_e\{h(t);\omega\} = \int_{-\infty}^{\infty} e^{i\omega t}\, h(t) dt$$

The original h(t) can be restored by Inverse Fourier Transform defined as

h(t) =$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} F_e(\omega) d\omega$
Fourier Transform of the Convolution
h(t)*g(t) =$\int_{-\infty}^{\infty} h(t-\tau) g(\tau) d\tau = \int_{-\infty}^{\infty} g(t-\tau) h(\tau) d\tau$ of the two functiona h(t) and g(t), which are defined in

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(-∞, ∞) is equal to the product of their Fourier Transform:
$F_e\{h(t)*g(t);\omega\} = H_e(\omega).G_e(\omega)$
Under the assumption that both $H_e(\omega)$ *and* $G_e(\omega)$ exist.
Another useful property of the Fourier Transform which is frequently used in solving applied problem is the Fourier Transform of derivatives of h(t).The Fourier Transform of n-th order derivative of h(t) is:
$F_e\{\{h^n(t);\omega\} = (-i\omega)^n H_e(\omega)$

We can easily evaluate exponential Fourier Transform of the Riemann-Liouville Derivatives, Riemann-Liouville Derivatives and Caputo's Fractional Derivatives with lower terminal $a = -\infty$

$$F_e\{D_t^\alpha g(t);\omega\} = (-i\omega)^{\alpha-n} F_e(g^n(t);\omega)$$
$$= (-i\omega)^{\alpha-n}(-i\omega)^n\, G_e(\omega)$$
$$= (-i\omega)^\alpha\, G_e(\omega)$$

Where D^α denotes any of the mentioned Fractional derivatives.

IV. CONCLUSION

In this paper, we deals with transform of Fractional Derivatives , namely, the Laplace Transform and Fourier Transform. and offer the corresponding formula for transform of fractional derivative . The formulation concerning Fourier and Laplace Transform as brought out in this paper are anticipated to be of a prototype consideration towards perspective developments for analysing the oscillation equation**[8]** with a fractional order damping term and for studying relaxation processes in insulators .Laplace Transform can be useful for solving applied problem leading to linear fractional differential equation with constant coefficient with accompanying initial condition in traditional form. We leave such considerations open for a future research.

V. REFERENCES

[1] [7] M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, Geophys. JR Astr. Soc. 13 (1967), 529–539.

[2] M. Caputo and F. Mainardi, A new dissipation model based on memory mechanism,Pure Appl. Geophys. 91 (1971), 134–147.

[3 L. Debnath, Integral Transforms and Their Applications, CRC Press, Florida,1995.

[4] --------------A brief historical introduction to fractional calculus, to appear in Internat.J. Math. Ed. Sci. Tech., 2003.

[5] The Fractional Calculus, Mathematics in Science and Engineering, vol.111, Academic Press, New York, 1974. [6] M. C. Potter, J. L. Goldberg, E. Aboufadel, "Advanced Engineering Mathematics", Oxford University Press; 3 edition, ISBN-13: 978-0195160185 (2005).

[7] L. Debnath, "Nonlinear Partial Differential Equations for Scientists and Engineers", Birkhäuser; 3rd edition, ISBN-13: 978-0817682644 (2012).

[8] P. Espanol, P. Warren, "Statistical Mechanics of Dissipative Particle Dynamics", Europhys. Lett. 30 191, Issue 4 (1995).

[9] K. S. Miller, "An Introduction to the Fractional Calculus and Fractional Differential Equations", Wiley-Interscience; 1 edition, ISBN-13: 978-0471588849 (1993).

[10] P.Williams, "Fractional Calculus of Schwartz Distributions", Department of Mathematics and Statistics, The University of Melbourn (2007).

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