Existence and Uniqueness of Solution of Linear Fractional Differential Equation

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Abstract: **In this paper we bring the proof of existence and uniqueness of solution of initial value problem of linear fractional differential equation and finally we solve some linear fractional differential equation**

Keyword: **Fractional Derivatives, Laplace Transform, Convolution of functions, existence and uniqueness of solution.**

I. INTRODUCTION

In recent years, fractional order differential equations have become very popular mathematical modeling [1]. A physical interpretation of fractional integral and derivative is given in [2]. Although there are many approach to generalize the nth derivative of f(t),but the most commonly used definitions are Riemann–Liouville and Caputo fractional derivatives. Fractional-order differential equations occur in a surprising number of real-world models. At the heart of a lot of applications is the phenomenon of anomalous diffusion. The isotropic normal diffusion equation is (with time scaled to remove physical constants):

$$u_t - \Delta u = 0$$

and can be derived in a number of ways: a random walk model, Fick's law of diffusion and the Langevin equation are discussed in[3]. According to Vlahos et al., the assumptions for these models are fair for diffusion in homogeneous media, but not for a medium which is highly heterogeneous, a particular case they discuss is when the diffusive system is far from equilibrium

II. OVERVIEW: FRACTIONAL CALCULUS

Fractional Calculus is a term used for the theory of derivatives and integrals of arbitrary order, which generalize the notion of integer order differentiation and n-fold integration. The idea behind Fractional calculus is to generalize the definition of differentiation and integration with order $n \in \mathbb{N}$ to order $s \in \mathbb{R}$. The first discussion [9] on Fractional Calculus began in 1695 in a letter to L'Hopital by Leibniz in which he discussed about calculus of arbitrary order. Fractional Calculus is three centuries old. Few names that laid the foundation of Fractional Calculus are Abel, Liouville, Riemann, Euler, Caputo etc. Fractional Calculus has recently been applied in various areas of engineering, science, finance, applied mathematics and bio engineering.[10] . It has earlier been observed that derivatives of non-integer order are useful for describing the properties of various real materials like polymer, rocks etc. Also the fractional order models were found more logical to talk an discuss about than the integer-order models. In this paper we are focusing on Fractional Derivatives. Different people gave different definitions for the Fractional Derivative. Few definitions are :

Grunwald-Letnikov Fractional Derivatives: Let us consider a continuous function f(t)

We define

$${}_aD_t^p f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh)$$

The above formula has been obtained under the assumption that the derivatives $f^{(k)}(t)$ (k=1, 2, 3,. . . ,m+1) are continuous in the closed interval [a,t] and that m is the integer number satisfying m > p-1.

Riemann-Liouville Derivatives:

$${}_aD_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \ (m \le p < m+1)$$

Caputo's Fractional Derivatives:

The definition of the fractional differentiation of the Riemann-
Liouville Derivatives type played an important role in the development of the theory of fractional derivatives and for its applications in pure mathematics. However, the demands of modern technology require a certain revision of well established mathematical approach .The Caputo approach provides an interpolation between an integer order derivatives:

$${}^cD^\alpha f(x) = \frac{1}{\Gamma(\alpha-n)} \int_a^x \frac{f^{(n)}(u)}{(x-u)^{(\alpha-n+1)}} \quad , \quad n-1 < \alpha < n \, , \alpha \in \mathbb{R} \, , n \in \mathbb{N}$$

Euler's Fractional Derivatives:

$$\frac{d^{\propto}}{dt^{\propto}}[t^{\beta}] = D_t^{\alpha}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \alpha \in \mathbb{R}$$

.

Sequential Fractional Derivatives:

The main idea of differentiation and integration of arbitrary order is the generalization of iterated integration and differentiation. In all these approaches we replace the integer valued parameter n of a operator denoted by $\frac{d^n}{dt^n}$ with a non integer parameter p.

However, we can assume that the n-th order differentiation is simply a series of n first order differentiation .So, considering more general expressions

$$D_t^{\alpha} = D_t^{\alpha_1} D_t^{\alpha_2} D_t^{\alpha_3} \dots\dots\dots . D_t^{\alpha_n}$$

Where $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \cdots \dots\dots . \alpha_n$,which we will also call the sequential fractional derivatives.

Indeed, Riemann-Liouville Derivatives can be written as

$${}_aD_t^p f(t) = \frac{d}{dt}\frac{d}{dt} \dots\dots . \frac{d}{dt}\, {}_aD_t^{-(n-p)} f(t) \quad (n-1 \leq p < n)$$

While the Caputo fractional differential operator can be written as

$${}^{C}D^{\alpha} f(x) = {}_aD_t^{-(n-p)} \frac{d}{dt} \dots\dots . \frac{d}{dt} f(t) \quad (n-1 < p \leq n-1)$$

Properties of Fractional Derivatives:

Fractional Derivatives satisfy almost all the properties that hold for[5] ordinary derivatives. We are aware of the general properties of the derivative operator $D_t^n, n \in \mathbb{N}$. Below mentioned are the properties of Fractional Derivative that can be easily verified:

- $D_t^{\propto}[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[g(t)]$ where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.
- $D_t^{\propto}[f(t)C] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[C] = D_t^{\propto}[f(t)]C$.
- $D_t^{\propto}[h(t)+g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[t^0] D_t^k[h(t)+g(t)] = D_t^{\propto}[h(t)] + D_t^{\propto}[g(t)]$.
- $D_t^{\propto}[h(at)] = a^{\propto} D_x^{\propto}[h(x)], x = at$.
- $D_t^{\propto}[t^{-m}] = (-1)^{\propto} \frac{\Gamma(m+\propto)}{\Gamma(m)} t^{-(m+\propto)}$.
- $D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}(f(t))] = D_t^{\nu}[D_t^{\mu}(f(t))]$.

$$D_t^{-1}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1},$$

Where

$\alpha \in D_t^{\propto}[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[g(t)]$,

where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.

- $D_t^{\propto}[f(t)C] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[C] = D_t^{\propto}[f(t)]C$ Where C is an arbitrary constant.
- $D_t^{\propto}[h(t)+g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[t^0] D_t^k[h(t)+g(t)] = D_t^{\propto}[h(t)] + D_t^{\propto}[g(t)]$.
- $D_t^{\propto}[h(at)] = a^{\propto} D_x^{\propto}[h(x)]$ under the scaling $x = at$.
- $D_t^{\propto}[t^{-m}] = (-1)^{\propto} \frac{\Gamma(m+\propto)}{\Gamma(m)} t^{-(m+\propto)}$ for a given $m \in \mathbb{R}$.
- $D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}(f(t))] = D_t^{\nu}[D_t^{\mu}(f(t))]$ under the composition of D_t^{ν} and D_t^{μ}on $f(t)$.
- $D_t^{-1}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$, where $\beta \in \mathbb{R}$ corresponding to a negative order derivative.

Mittag-Leffler Function:

The Exponential function play a important role in the theory of integer order differential equation its one parameter generalization is denoted by[4]

$$\boldsymbol{E_{\alpha}(z)} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

was introduced by G.M Mittag Leffler [5, 6, 7] and also studied by A.William[8, 9] **.**

III. LAPLACE TRANSFORMS AND INVERSE LAPLACE OF FRACTIONAL DERIVATIVES:

The Laplace transform of a function f(t) is defined as

$$F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt$$

For existence of Laplace transform of f(t), f(t) must be of exponential order. The original f(t) can be obtained from F(s) with the help of [3]Inverse Laplace Transform

$$f(t) = L^{-1}[F(s), t] = \int_{c-i\infty}^{c+i\infty} e^s F(s) ds , \quad c = Re(s) > c_0$$

Where c_0 lies in right half plane of the absolute convergence of Laplace integral.

Laplace Transform of convolution is defined as

$$f(t)*g(t) = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t g(t-\tau) f(\tau) d\tau$$

Another useful property which we are needed is Laplace Transform of derivative of an integer order n of a function f(t) :

$$L\{f^n(t);\ s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} \{f^n(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0)$$

Likewise, we can easily prove Laplace Transform of Fractional derivatives of order p> 0 in terms of Riemann-Liouville Derivatives p[th]

$$L\{{}_0D_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k [{}_0D_t^{p-k-1} f(t)] \quad (n-1 \leq p < n)$$

Similarly, we can easily establish Laplace Transform Caputo Derivative as

$$L\{{}^{C}D^{\alpha} f(x); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} \{f^n(0)\} \quad (n-1 < p \leq n)$$

IV. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this chapter we consider the existence and uniqueness of solutions of initial value problem of fractional order differential equation. First we consider the case of linear fractional order differential equations with continuous coefficients and bring the proof of existence and uniqueness theorem for one –term and n-term fractional differential equations.

Then we give the proof of existence and uniqueness theorem for general term fractional differential equations. Finally, we discuss the dependence of solution of general fractional differential equations on initial conditions.

Linear Fractional Differential Equation.

In this section the existence and uniqueness of solutions of initial value problem for liner fractional differential equations with sequential derivatives are discussed.

Let's consider the following initial value problem:

$$ {}_0D_t^{\sigma_n}y(t) + \sum_{j=1}^{n-1} p_j(t)\,[{}_0D_t^{\sigma_{n-j}}y(t)] + p_n(t)y(t) = f(t) $$

$(0< t < T <\propto)$ (4.1.1)

$[{}_0D_t^{\sigma_{n-1}}y(t)]_{t=0}= b_k$,k=1,2,3…………………………………..n, (4.1.2)

Where ${}_\alpha D_t^{\sigma_k} = {}_\alpha D_t^{\alpha_k}\,{}_\alpha D_t^{\alpha_{k-1}} \ldots\ldots\ldots\,{}_\alpha D_t^{\alpha_1}$ ${}_\alpha D_t^{\uparrow\sigma_{k-1}} = {}_\alpha D_t^{\uparrow\alpha_{k-1}}\,{}_\alpha D_t^{\alpha_{k-1}} \ldots\ldots\ldots\ldots\ldots\ldots\,{}_\alpha D_t^{\alpha_1}$

Where $\sigma_k = \sum_{j=i}^{k}\alpha_j$ (k=1.2…………………..n)

$0< \alpha_j \le 1 \ (j = 1,2 \ldots\ldots\ldots..n)$

And f(t) ∈ $L_1(0,T)$ i.e. $\int_0^T |f(t)| < \infty$

Here we assume f(t) = 0 for t > T. Also we can have p_k(t) = 0 for k =1,2………n

Theorem:

If $(t) \in L_1(0,T)$, Then the equation ${}_0D_t^{\sigma_n}y(t) = f(t)$ (4.1.1.1)

has unique solution $y(t) \in L_1(0,T)$, which satisfying the initial conditions given by (4.1.2)

Proof: Using Laplace transform of Sequential Fractional Derivative and equation (4.1.1.1), we get

s^{σ_n}Y(s) $+\sum_{k=0}^{n-1} s^{\alpha_n-\sigma_{n-k}}[\,[{}_0D_t^{\sigma_{n-k-1}}y(t)]_{t=0}$= F(s)

Where,Y(s) and F(s) denote the Laplace transform of y(t) and f(t).

Using Initial Conditions:

Y(s) $=s^{-\sigma_n}$ F(s) $+\sum_{k=0}^{n-1} b_{n-k} s^{-\sigma_{n-k}}$ and by Inverse Laplace Transform

y(t)=1/Γ(σ_n) $\int_0^t (t-\tau)^{\sigma_n-1} f(\tau)d\tau + \sum_{k=0}^{n-1}\frac{b_{n-k}}{\Gamma(\sigma_{n-k})} t^{\sigma_{n-k-1}}$

For $n-k=i$ we have y(t)=1/Γ(σ_n) $\int_0^t (t-\tau)^{\sigma_n-1} f(\tau)d\tau + \sum_{i=1}^{n}\frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i-1}$

Now by definition of Riemann-Liouville Derivatives of power function and taking in to account that $\frac{1}{\Gamma(-m)} = 0$ for m = 1,2,3 … …. we can easily obtained

${}_0D_t^{\sigma_k}\left(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}\right) = \frac{t^{\sigma_i-\sigma_k-1}}{\Gamma(\sigma_i-\sigma_k)}$, k< i and ${}_0D_t^{\sigma_k}\left(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}\right) = 0$ whenk ≥ i

${}_0D_t^{\sigma_k}\left(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}\right) = \frac{t^{\sigma_i-\sigma_k}}{\Gamma(\sigma_i-\sigma_k-1)}$, k< i and ${}_0D_t^{\sigma_k}\left(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}\right) = 1$ when i = k

${}_0D_t^{\sigma_k}\left(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}\right) = 0$ if k> i Where k = 1, 2,………n and i = 1,2,3…………………..n

It follows that y(t) ∈ $L_1(0,T)$ and it satisfies the initial conditions. So existence of solution is proved. Uniqueness follows from the linear property of fractional derivative and Laplace Transform.

Indeed If there exist two solutions $y_1(t)$and $y_2(t)$ of considered problem, then the function

$z(t) = y_1(t) - y_2(t)$ must satisfies the ${}_0D_t^{\sigma_n}z(t) = 0$ and the initial conditions which gives Laplace transform of z(t) as zero and it proves the uniqueness of the solution.

Theorem: If $f(t) \in L_1(0,T)$ and $p_j(t)$ (j=1,2,3……………… n) are continuous functions in the closed interval [0,T] Then the initial value problem (4.1.1-4.1.2) has a unique solution $y(t) \in L_1(0,T)$.

Proof: Let us assume that the above equation has solution y(t) and, let us consider

${}_0D_t^{\sigma_n}y(t) = \varphi(t)$

Using previous theorem

y(t) = 1/Γ(σ_n) $\int_0^t (t-\tau)^{\sigma_n-1} f(\tau)d\tau + \sum_{i=1}^{n}\frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i-1}$ (4.1.2.1)

using 4.1.1 and 4.1.2.1 we have

$$ {}_0D_t^{\sigma_n}y(t) + \sum_{k=1}^{n-1} p_{n-k}(t)\,{}_0D_t^{\sigma_k}y(t) + p_n(t)y(t) = f(t) $$

We obtain the volterra integral equation for the function $\varphi(t)$:

$$ \varphi(t) + \int_0^t K(t,\tau)\ \varphi(t)dt = g(t) $$

Where $K(t,\tau) = p_n(t)\frac{(t-\tau)^{\sigma_n-1}}{\Gamma(\sigma_n)} + \sum_{k=1}^{n-1} p_{n-k}(t)\frac{(t-\tau)^{\sigma_n-\sigma_k-1}}{\Gamma(\sigma_n-\sigma_k)}$

$$ g(t) = f(t) - p_n(t)\sum_{i=1}^{n} b_i\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)} - \sum_{k=1}^{n-1} p_{n-k}(t)\sum_{i=k+1}^{n} b_i\frac{t^{\sigma_i-\sigma_k-1}}{\Gamma(\sigma_i-\sigma_k)} $$

As $p_j(t)$ (j=1,2,3……………… n) are continous functions in the closed interval [0,T]

So we have

$K(t,\tau) = \frac{k^*(t,\tau)}{(t-\tau)^{1-\mu}}$ and

Where $k^*(t, \tau)$ is continous for $0 \le t \le T$ and $0 \le \tau \le T$ and

$$\mu = \min(\sigma_n, \sigma_n - \sigma_{n-1}, \sigma_n - \sigma_{n-2}, \sigma_n - \sigma_{n-3} \dots \dots \dots \dots \dots \dots \dots \dots \dots . . \sigma_n - \sigma_1) = \min(\sigma_n, \alpha_n)$$

Similarly, g(t) can be written

$$g(t) = \frac{g^*(t)}{(t)^{1-\nu}}$$

Where $g^*(t)$ is continous in [0,T] and

$\nu = \min(\sigma_1, \sigma_2, \sigma_3 \dots . . \sigma_n; \sigma_2 - \sigma_1, \dots \dots \dots \dots \dots \dots \sigma_n - \sigma_1; \dots \dots . . \sigma_n - \sigma_{n-1}) = \min(\sigma_n, \alpha_n)$

clearly $0 < \mu \le 1$ and $0 < \nu \le 1$ .It is known that the equation with weak singular kernel and with choice of g(t) has a unique solution $\varphi(t) \in L_1(0, T)$.The unique solution $(t) \in L_1(0, T)$ can be obtained using the previous theorem

5. Examples

In this section We use Laplace Method to solve ordinary Fractional differential Equation.

Example1: Consider the equation

$${}_0D_t^{\frac{1}{2}} f(t) + af(t) = 0 \quad (t > 0);$$

$[{}_0D_t^{\frac{-1}{2}} f(t)]_{t=0} = C$

Applying Laplace Transform, we get

$$F(S) = \frac{C}{s^{1/2} + a}, \quad [{}_0D_t^{\frac{-1}{2}} f(t)]_{t=0} = C$$

Using Inverse Laplace Transform

$$f(t) = Ct^{1/2} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})$$

Example2: Let us consider the following Equations

$${}_0D_t^Q f(t) + D_t^q f(t) = h(t)$$

Let's assume here $0 < q < Q < 1$. Laplace Transform Of above equations leads to

$((s^Q + s^q)F(s) = C + H(s)$

$[{}_0D_t^{Q-1} f(t) + D_t^{q-1} f(t)]_{t=0} = C$ and then after taking inversion for $\beta = Q$ and $\alpha = Q - q$

we get

$$f(t) = CG(t) + \int_0^t G(t - \tau)h(\tau)d\tau$$

$$[{}_0D_t^{Q-1} f(t) + D_t^{q-1} f(t)]_{t=0} = C$$

$$G(t) = t^{Q-1} \boldsymbol{E}_{\boldsymbol{Q-q,Q}}(-\boldsymbol{t}^{\boldsymbol{Q-q}})$$

Example 3: Consider the following non-homogenous fractional differential equation

$${}_0D_t^\alpha y(t) + \lambda y(t) = h(t)$$

$[{}_0D_t^{\alpha-k} y(t)]_{t=0} = b_k$,k=1,2,3……………………………………..n, where $n - 1 < \alpha < n$

Taking into account the initial conditions ,The Laplace Transform of the above equation is

$$s^\alpha Y(s) - \lambda Y(s) = H(s) + \sum_{k=1}^{n} b_k s^{k-1}$$

The inverse Laplace Transform give the solution:

$$Y(t)y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} \boldsymbol{E}_{\boldsymbol{\alpha,\alpha-k+1}}(\lambda \boldsymbol{t}^{\boldsymbol{\alpha}}) + \int_0^t (t - \tau)^{\alpha-1} \boldsymbol{E}_{\boldsymbol{\alpha,\alpha}}(\lambda(t - \tau)^\alpha) h(\tau) d\tau.$$

V. CONCLUSION

In this paper, we prove the existence and uniqueness of solutions of initial value problem of fractional order differential equation. First we consider the case of linear fractional order differential equations with continuous coefficients and bring the proof of existence and uniqueness theorem for one –term and n-term fractional differential equations.Then We Illustrate some examples using Laplace Transform Method to solve Ordinary Fractional Differential equations..

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