A New Notion of b*ĝ-Closed Sets in Topological Spaces

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Abstract- **The determination of this paper is to define b*ĝ-limit point, b*ĝ-derived set, b*ĝ-border, b*ĝ-frontier and b*ĝ-exterior of a subset of a topological space using the concept of b*ĝ-open sets and study some of their properties.**

Keywords: b*ĝ-border, b*ĝ-closed, b*ĝ-closure, b*ĝ-derived set, b*ĝ-exterior, b*ĝ-frontier, b*ĝ-interior, b*ĝ-limit point, and b*ĝ-open.

1. INTRODUCTION

In 1973, Das [4] defined semi-interior point and semi-limit point of a subset. The semi-derived set of a subset of a topological space was also defined and studied by him. In 2016, K.Bala Deepa Arasi and G.Subasini [1] introduced b*ĝ-closed sets and studied some of its properties. In 2017, we [2] introduced b*ĝ-continuous functions and b*ĝ-open maps. Further some of their basic properties are studied and compared with the other known existing functions. Also, in 2017, we [3] introduced Contra b*ĝ-continuous functions and its properties are discussed. Also, we relate this function with the other known existing functions.

Now, we define a new class of sets namely b*ĝ-limit points, b*ĝ-derived sets, b*ĝ-border, b*ĝ-frontier and b*ĝ-exterior of a subset of a topological space and studied some of their properties. Also, we prove some of the properties of b*ĝ-closure and b*ĝ-interior of a subset of a topological space.

2.PRELIMINARIES

Throughout this paper (X,τ) (or simply X) represents topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X,τ), $Cl(A)$, $Int(A)$, $D(A)$, $b(A)$ and $Ext(A)$ denote the closure, interior, derived, border and exterior of A respectively. We are giving some basic definitions.

Definition 2.1: [1] A subset A of a topological space (X,τ) is called

1) ***b*ĝ-closed*** set if b*cl(A) ⊆ U whenever $A \subseteq U$ and U is ĝ-open in X. The collection of all b*ĝ-closed sets in (X,τ) is denoted by b*ĝ-C(X,τ).
2) ***b*ĝ-open*** set if $X\setminus A$ is b*ĝ-closed in A. The collection of all b*ĝ-open sets in (X,τ) is denoted by b*ĝ-O(X,τ).

Definition 2.2: Let A be the subset of a space (X,τ). Then

1) The ***border*** of A is defined as $b(A) = A \setminus Int(A)$.
2) The ***frontier*** of A is defined as $Fr(A) = Cl(A) \setminus Int(A)$.
3) The ***exterior*** of A is defined as $Ext(A) = Int(X \setminus A)$.

Theorem 2.3:[1]

1) Every closed set is b*ĝ-closed.
2) Every open set is b*ĝ-open.

3. Properties of b*ĝ-interior and b*ĝ-closure

Definition 3.1: The ***b*ĝ-interior*** of A is defined as the union of all b*ĝ-open sets of X contained in A. It is denoted by b*ĝ *Int*(A).

Definition 3.2: A point x ∈ X is called ***b*ĝ-interior point*** of A if A contains a b*ĝ-open set containing x.

Definition 3.3: The ***b*ĝ-closure*** of A is defined as the intersection of all b*ĝ-closed sets of X containing A. It is denoted by b*ĝ *Cl*(A).

Theorem 3.4: If A is a subset of X, then b*ĝ *Int*(A) is the set of all b*ĝ-interior points of A.

Proof: If $x \in$ b*ĝ *Int*(A), then x belongs to some b*ĝ-open subset U of A. That is, x is a b*ĝ-interior point of A.

Remark 3.5: If A is any subset of X, b*ĝ *Int*(A) is b*ĝ-open. In fact b*ĝ *Int*(A) is the largest b*ĝ-open set contained in A.

Remark 3.6: A subset A of X is b*ĝ-open ⇔ b*ĝ *Int*(A) = A.

Result 3.7: For the subset A of a topological space (X,τ), *Int*(A) ⊆ b*ĝ *Int*(A).

Proof: Since *Int*(A) is the union of open sets and by theorem 2.3 (2), *Int*(A) is b*ĝ-open. It is clear from the definition 3.1 that $Int(A) \subseteq b^*\hat{g}\,Int(A)$.

Theorem 3.8: Let A and B be the subsets of a topological space (X,τ), then the following result holds:

1) $b^*\hat{g}\,Int(\Phi) = \Phi$;
2) $b^*\hat{g}\,Int(X) = X$;
3) $b^*\hat{g}\,Int(A) \subseteq A$;
4) $A \subseteq B \Longrightarrow b^*\hat{g}\,Int(A) \subseteq b^*\hat{g}\,Int(B)$;
5) $b^*\hat{g}\,Int(A \cup B) \supseteq b^*\hat{g}\,Int(A) \cup b^*\hat{g}\,Int(B)$;
6) $b^*\hat{g}\,Int(A \cap B) \subseteq b^*\hat{g}\,Int(A) \cap b^*\hat{g}\,Int(B)$;
7) $b^*\hat{g}\,Int(Int(A)) = Int(A)$;
8) $Int(b^*\hat{g}\,Int(A)) \subseteq Int(A)$;
9) $b^*\hat{g}\,Int(b^*\hat{g}\,Int(A)) = b^*\hat{g}\,Int(A)$;

Proof: (1), (2) and (3) follows from definition 3.1.

(4) From definition 3.1 we have, $b^*\hat{g}\,Int(A) \subseteq A$. Since $A \subseteq B$, $b^*\hat{g}\,Int(A) \subseteq B$. But $b^*\hat{g}\,Int(B) \subseteq B$. By remark 3.5, $b^*\hat{g}\,Int(A) \subseteq b^*\hat{g}\,Int(B)$.

(5) Since $A \subseteq A \cup B$; $B \subseteq A \cup B$ and by (4) we have, $b^*\hat{g}\,Int(A) \subseteq b^*\hat{g}\,Int(A \cup B)$ and $b^*\hat{g}\,Int(B) \subseteq b^*\hat{g}\,Int(A \cup B)$. Therefore $b^*\hat{g}\,Int(A) \cup b^*\hat{g}\,Int(B) \subseteq b^*\hat{g}\,Int(A \cup B)$.

(6) Since $A \cap B \subseteq A$; $A \cap B \subseteq B$ and by (4) we have, $b^*\hat{g}\,Int(A \cap B) \subseteq b^*\hat{g}\,Int(A)$ and $b^*\hat{g}\,Int(A \cap B) \subseteq b^*\hat{g}\,Int(B)$. Therefore $b^*\hat{g}\,Int(A \cap B) \subseteq b^*\hat{g}\,Int(A) \cap b^*\hat{g}\,Int(B)$.

(7) Since *Int*(A) is an open set and by theorem 2.3 (2), *Int*(A) is b*ĝ-open. By remark 3.6, $b^*\hat{g}\,Int(Int(A)) = Int(A)$.

(8) From definition 3.1 we have, $b^*\hat{g}\,Int(A) \subseteq A$. Clearly, it follows that $Int(b^*\hat{g}\,Int(A)) \subseteq Int(A)$;

(9) Follows from remark 3.6 and 3.5.

Remark 3.9: If A is any subset of X, $b^*\hat{g}\,Cl(A)$ is b*ĝ-closed. In fact $b^*\hat{g}\,Cl(A)$ is the smallest b*ĝ-closed set containing A.

Remark 3.10: A subset A of X is b*ĝ-closed $\Leftrightarrow$ $b^*\hat{g}\,Cl(A) = A$.

Theorem 3.11: Let A and B be the subsets of a topological space (X,τ), then the following result holds:

1) $b^*\hat{g}\,Cl(\Phi) = \Phi$;
2) $b^*\hat{g}\,Cl(X) = X$;
3) $A \subseteq b^*\hat{g}\,Cl(A)$;
4) $A \subseteq B \Longrightarrow b^*\hat{g}\,Cl(A) \subseteq b^*\hat{g}\,Cl(B)$;
5) $b^*\hat{g}\,Cl(b^*\hat{g}\,Cl(A)) = b^*\hat{g}\,Cl(A)$;
6) $b^*\hat{g}\,Cl(A \cup B) \supseteq b^*\hat{g}\,Cl(A) \cup b^*\hat{g}\,Cl(B)$;
7) $b^*\hat{g}\,Cl(A \cap B) \subseteq b^*\hat{g}\,Cl(A) \cap b^*\hat{g}\,Cl(B)$;
8) $b^*\hat{g}\,Cl(Cl(A)) = Cl(A)$;
9) $Cl(b^*\hat{g}\,Cl(A)) = Cl(A)$;

Proof: (1), (2) and (3) follows from definition 3.3.

(4) From definition 3.3 we have, $A \subseteq b^*\hat{g}\,Cl(A)$. Since $A \subseteq B$, $b^*\hat{g}\,Cl(A) \subseteq B$. But $b^*\hat{g}\,Cl(B)$ is the smallest b*ĝ-closed set in X containing B. Therefore $b^*\hat{g}\,Cl(A) \subseteq b^*\hat{g}\,Cl(B)$.

(5) Follows from remark 3.9 and 3.10.

(6) Since $A \subseteq A \cup B$; $B \subseteq A \cup B$ and by (4) we have, $b^*\hat{g}\,Cl(A) \subseteq b^*\hat{g}\,Cl(A \cup B)$ and $b^*\hat{g}\,Cl(B) \subseteq b^*\hat{g}\,Cl(A \cup B)$. Therefore $b^*\hat{g}\,Cl(A) \cup b^*\hat{g}\,Cl(B) \subseteq b^*\hat{g}\,Cl(A \cup B)$.

(7) Since $A \cap B \subseteq A$; $A \cap B \subseteq B$ and by (4) we have, $b^*\hat{g}\,Cl(A \cap B) \subseteq b^*\hat{g}\,Cl(A)$ and $b^*\hat{g}\,Cl(A \cap B) \subseteq b^*\hat{g}\,Cl(B)$. Therefore $b^*\hat{g}\,Cl(A \cap B) \subseteq b^*\hat{g}\,Cl(A) \cap b^*\hat{g}\,Cl(B)$.

(8) Since $Cl(A)$ is a closed set and by theorem 2.3 (1), $Cl(A)$ is b*ĝ-closed. Therefore by remark 3.10, $b^*\hat{g}\,Cl(Cl(A)) = Cl(A)$.

(9) Follows from remark 3.9 and 3.10.

4. Applications of b*ĝ-Open Sets

Definition 4.1: Let A be a subset of a topological space X. A point $x \in X$ is said to be ***b*ĝ-limit point*** of A if for every b*ĝ-open set U containing x, $U \cap (A \setminus \{x\}) \neq \Phi$. The set of all b*ĝ-limit points of A is called an ***b*ĝ-derived set*** of A and is denoted by $b^*\hat{g}\,(A)$.

Example 4.2: Let $X = \{a,b,c\}$ with the topology $\tau = \{X,\Phi,\{a\},\{c\},\{b,c\},\{a,c\}\}$ and $b^*\hat{g}\ O(X) = \{X,\Phi,\{a\},\{c\},\{a,c\},\{b,c\}\}$. If $A = \{c\}$, then $b^*\hat{g}\,(A) = \{b\}$.

Result 4.3: Let A be a subset of a topological space X. Then,
(i) $b^*\hat{g}\,Cl(X \setminus A) = X \setminus b^*\hat{g}\,Int(A)$
(ii) $b^*\hat{g}\,Int(X \setminus A) = X \setminus b^*\hat{g}\,Cl(A)$

Proof: (i) Let $x \in X \setminus b^*\hat{g}\,Int(A)$. Then, $x \notin b^*\hat{g}\,Int(A)$. This implies that x does not belongs to any b*ĝ-open subset of A. Let F be a b*ĝ-closed set containing $X \setminus A$. Then $X \setminus F$ is b*ĝ-open set contained in A. Therefore, $x \notin X \setminus F$ and so $x \in F$. Hence, $x \in b^*\hat{g}\,Cl(X \setminus A)$. This implies $X \setminus b^*\hat{g}\,Int(A) \subseteq b^*\hat{g}\,Cl(X \setminus A)$. On the other hand, let $x \in b^*\hat{g}\,Cl(X \setminus A)$. Then x belongs to every b*ĝ-closed set containing $X \setminus A$. Hence, x

does not belongs to any b*ĝ-open subset of A. That is $x \notin$ b*ĝ Int(A). This implies $x \in X\setminus$ b*ĝ Int(A). Therefore, b*ĝ $Cl(X\setminus$A) $\subseteq X \setminus$ b*ĝ Int(A). Thus, b*ĝ $Cl(X\setminus$A) = $X\setminus$ b*ĝ Int(A).

(ii) can be proved by replacing A by $X\setminus$A in (i) and using set theoretic properties.

Theorem 4.4: For subsets A, B of a space X, the following statement holds:

1) $D(A) \subseteq$ b*ĝ $D(A)$, where $D(A)$ is the derived set of A;
2) b*ĝ $D(\Phi) = \Phi$;
3) If $A \subset B$, then b*ĝ $D(A) \subseteq$ b*ĝ $D(B)$;
4) b*ĝ $D(A \cup B) \supseteq$ b*ĝ $D(A) \cup$ b*ĝ $D(B)$;
5) b*ĝ $D(A \cap B) \subseteq$ b*ĝ $D(A) \cap$ b*ĝ $D(B)$;
6) b*ĝ $D(A) \subseteq$ b*ĝ $D(A\setminus\{x\})$;

Proof: (1) Let $x \in D(A)$. By the definition of $D(A)$, there exist an open set U containing x such that $U \cap (A\setminus\{x\}) \neq \Phi$. By theorem 2.3(2), U is an b*ĝ-open set containing x such that $U \cap (A\setminus\{x\}) \neq \Phi$. Therefore, $x \in$ b*ĝ (A). Hence, $D(A) \subseteq$ b*ĝ (A).

(2) For all b*ĝ-open set U and for all $x \in X$, $U \cap (\Phi \setminus\{x\}) = \Phi$. Hence, b*ĝ $(\Phi) = \Phi$.

(3) Let $x \in$ b*ĝ (A). Then for each b*ĝ-open set U containing x, $U \cap (A\setminus\{x\}) \neq \Phi$. Since $A \subseteq B$, $U \cap (B\setminus\{x\}) \neq \Phi$. This implies that $x \in$ b*ĝ (B). Hence, b*ĝ $(A) \subseteq$ b*ĝ (B).

(4) Let $x \in$ b*ĝ $(A) \cup$ b*ĝ (B). Then $x \in$ b*ĝ (A) or $x \in$ b*ĝ (B). If $x \in$ b*ĝ (A), then for each b*ĝ-open set U containing x, $U \cap (A\setminus\{x\}) \neq \Phi$. Since $A \subseteq A \cup B$, $U \cap (A \cup B\setminus\{x\}) \neq \Phi$. This implies that $x \in$ b*ĝ $(A \cup B)$. Hence, b*ĝ $(A) \subseteq$ b*ĝ $(A \cup B)$ ……….. (1). Otherwise, if $x \in$ b*ĝ (B), then for each b*ĝ-open set U containing x, $U \cap (B\setminus\{x\}) \neq \Phi$. Since $B \subseteq A \cup B$, $U \cap (A \cup B\setminus\{x\}) \neq \Phi$. This implies that $x \in$ b*ĝ $(A \cup B)$. Hence, b*ĝ $(B) \subseteq$ b*ĝ $(A \cup B)$ ……….. (2). From (1) and (2), b*ĝ $(A) \cup$ b*ĝ $(B) \subseteq$ b*ĝ $(A \cup B)$.

(5) Let $x \in$ b*ĝ $(A \cap B)$. Then for each b*ĝ-open set U containing x, $U \cap (A \cap B\setminus\{x\}) \neq \Phi$. Since $A \cap B \subseteq A$, $U \cap (A\setminus\{x\}) \neq \Phi$. This implies that $x \in$ b*ĝ (A). Also, since $A \cap B \subseteq B$, $U \cap (B\setminus\{x\}) \neq \Phi$. This implies that $x \in$ b*ĝ (B). Therefore, $x \in$ b*ĝ $(A) \cap$ b*ĝ (B). Thus, b*ĝ $(A \cap B) \subseteq$ b*ĝ$(A) \cap$b*ĝ$D(B)$.

(6) Let $x \in$ b*ĝ (A). Then for each b*ĝ-open set U containing x, $U \cap (A\setminus\{x\}) \neq \Phi$. This implies that $U \cap ((A\setminus\{x\})\setminus\{x\}) \neq \Phi$. This implies $x \in$ b*ĝ $(A\setminus\{x\})$. Hence, b*ĝ $(A) \subseteq$ b*ĝ $(A\setminus\{x\})$.

Definition 4.5: If A is a subset of X, then the ***b*ĝ-border*** of A is defined by b*ĝ $(A) = A \setminus$ b*ĝ $Int(A)$.

Theorem 4.6: For a subset A of a space X, the following statement holds:

1) b*ĝ $b(\Phi) = \Phi$;
2) b*ĝ $b(X) = \Phi$;
3) b*ĝ $b(A) \subseteq A$;
4) b*ĝ$b(A) \subseteq b(A)$, where $b(A)$ denotes the border of A;
5) b*ĝ $Int(A) \cup$ b*ĝ $b(A) = A$;
6) b*ĝ $Int(A) \cap$ b*ĝ $b(A) = \Phi$;
7) b*ĝ b(b*ĝ $Int(A)) = \Phi$;
8) b*ĝ Int(b*ĝ $b(A)) = \Phi$;
9) b*ĝ b(b*ĝ $b(A))$ = b*ĝ $b(A)$;
10) b*ĝ b(b*ĝ Cl$(A)) = \Phi$;
11) b*ĝ Cl(b*ĝ $b(A)) = \Phi$.

Proof: (1), (2) and (3) follows from definition 4.5.

(4) Let $x \in$ b*ĝ (A). Then by definition 4.5, $x \in A\setminus$b*ĝ $Int(A)$. This implies that $x \in A$ and $x \notin$ b*ĝ $Int(A)$. By result 3.7, $x \in A$ and $x \notin Int(A)$. This implies that $x \in A \setminus Int(A)$. This implies that $x \in b(A)$. Hence, b*ĝ $(A) \subseteq (A)$.

(5) and (6) follows from definition 5.5.

(7)b*ĝ b(b*ĝ $Int(A))$ = b*ĝ $Int(A) \setminus$ b*ĝInt(b*ĝ$Int(A))$ = b*ĝ $Int(A)\setminus$b*ĝ $Int(A)$ (by theorem 3.8(9)) which is Φ. Hence, b*ĝ (b*ĝ $Int(A)) = \Phi$.

(8) Let x ∈ b*ĝ Int(b*ĝ $b(A))$. By theorem 3.8 (3), x ∈ b*ĝ $b(A)$. On the other hand, since b*ĝ $(A) \subseteq A$, we have x ∈ b*ĝ $Int(A)$. Therefore, x ∈ b*ĝ $(A) \cap$ b*ĝ $Int(A)$ which is a contradiction to (6). Hence, b*ĝ Int(b*ĝ $b(A)) = \Phi$.

(9) b*ĝ b(b*ĝ$b(A))$ = b*ĝ $b(A)\setminus$ b*ĝInt(b*ĝ $b(A))$ = b*ĝ $b(A)\setminus\Phi$ = b*ĝ $b(A)$ (from (8)). Hence, b*ĝ (b*ĝ b $(A))$ = b*ĝ (A).

(10) b*ĝ b(b*ĝ Cl$(A))$ = b*ĝ Cl$(A)\setminus$ b*ĝ Int(b*ĝ Cl$(A))$ $\subseteq$ b*ĝ Cl$(A)\setminus$ b*ĝ Cl(A) (by (6)) = Φ.

(11) b*ĝ Cl(b*ĝ $b(A))$ = b*ĝ Cl$((A \setminus$b*ĝ $Int(A))$ $\subseteq$ b*ĝ Cl$((A\setminus A)$ (by (6)) = b*ĝ Cl$(\Phi) = \Phi$ (by theorem 3.11(1)).

Definition 4.7: If A is a subset of X, then the ***b*ĝ-frontier*** of A is defined by b*ĝ (A) = b*ĝ $C(A) \setminus$ b*ĝ $Int(A)$.

Theorem 4.8: Let A be a subset of a space X. Then the following statement holds:

1) b*ĝ $Fr(\Phi) = \Phi$;
2) b*ĝ $Fr(X) = \Phi$;
3) b*ĝ $Fr(A) \subseteq$ b*ĝ $Cl(A)$;
4) b*ĝ$Cl(A)$ = b*ĝ$Int(A) \cup$ b*ĝ $Fr(A)$;
5) b*ĝ $Int(A) \cap$ b*ĝ $Fr(A) = \Phi$;
6) b*ĝ $b(A) \subseteq$ b*ĝ $Fr(A)$;
7) b*ĝ Fr(b*ĝ $Int(A)) \subseteq$ b*ĝ $Fr(A)$;
8) b*ĝ Cl(b*ĝ $Fr(A)) \subseteq$ b*ĝ $Cl(A)$;
9) b*ĝ $Int(A) \subseteq$ b*ĝ $Cl(A)$;
10) b*ĝ Int(b*ĝ $Fr(A)) \subseteq$ b*ĝ $Cl(A)$;
11) b*ĝ Fr(b*ĝ $Fr(A)) = \Phi$;
12) X = b*ĝ $Int(A) \cup$ b*ĝ $Int(X\setminus A) \cup$ b*ĝ $Fr(A)$;
13) b*ĝ $Fr(A)$ = b*ĝ $Cl(A) \cap$ b*ĝ $Cl(X\setminus A)$;
14) b*ĝ $Fr(A)$ = b*ĝ $Fr(X\setminus A)$.

Proof: (1), (2), (3) and (4) follows from definition 4.7.

(5) b*ĝ $Int(A)$ ∩ b*ĝ $Fr(A)$ = b*ĝ $Int(A)$ ∩ (b*ĝ $Cl(A)$\b*ĝ $Int(A)$) ⊆ A ∩ (b*ĝ $Cl(A)$\A) (by theorem 3.8(3)). b*ĝ $Int(A)$ ∩ b*ĝ $Fr(A)$ ⊆ b*ĝ $Cl(A)$ ∩ (b*ĝ $Cl(A)$\b*ĝ $Cl(A)$) (by theorem 3.11(3)). b*ĝ $Int(A)$ ∩ b*ĝ $Fr(A)$ = b*ĝ $Cl(A)$ ∩ Φ = Φ. Hence, b*ĝ $Int(A)$ ∩ b*ĝ $Fr(A)$ = Φ.

(6) Let x ∈ b*ĝ (A). Then x ∈ A \ b*ĝ $Int(A)$. By theorem 3.11(3), x ∈ b*ĝ (A) \ b*ĝ $Int(A)$ = b*ĝ $Fr(A)$. Hence, b*ĝ (A) ⊆ b*ĝ (A).

(7) b*ĝ Fr(b*ĝ $Int(A)$) = b*ĝ Cl(b*ĝ $Int(A)$)\ b*ĝ Int(b*ĝ $Int(A)$) ⊆ b*ĝ $Cl(A)$ \ b*ĝ $Int(A)$ (by theorem 3.8 (3), (9)) which is b*ĝ $Fr(A)$. Hence, b*ĝ (b*ĝ $Int(A)$) ⊆ b*ĝ $Fr(A)$.

(8) From (3) we have, b*ĝ (b*ĝ $Fr(A)$) ⊆ b*ĝ Cl(b*ĝ $Cl(A)$) = b*ĝ $Cl(A)$ (by theorem 3.11(5)). Hence, b*ĝ (b*ĝ $F(A)$) ⊆ b*ĝ $Cl(A)$.

(9) follows from (4).

(10) From (9), b*ĝ Int(b*ĝ $Fr(A)$) ⊆ b*ĝ Cl(b*ĝ $Fr(A)$) ⊆ b*ĝ $Cl(A)$ (from (8)). Hence, b*ĝ Int(b*ĝ $Fr(A)$) ⊆ b*ĝ $Cl(A)$.

(11) b*ĝ Fr(b*ĝ $Fr(A)$) = b*ĝ Cl(b*ĝ $Fr(A)$) \ b*ĝ Int(b*ĝ $Fr(A)$) ⊆ b*ĝ $Cl(A)$ \b*ĝ $Cl(A)$ = Φ (from (8), (10)). Hence, b*ĝ (b*ĝ $Fr(A)$) = Φ.

(12) b*ĝ $Int(A)$ ∪ b*ĝ $Int(X\setminus A)$ ∪ b*ĝ $Fr(A)$ = b*ĝ $Cl(A)$ ∪ b*ĝ $Int(X\setminus A)$ (from (4)) = b*ĝ $Cl(A)$ ∪ {X\b*ĝ $Cl(A)$} (by result 4.3 (ii)) which is X. Hence, X = b*ĝ $Int(A)$ ∪ b*ĝ $Int(X\setminus A)$ ∪ b*ĝ $Fr(A)$.

(13) b*ĝ $Cl(A)$ ∩ b*ĝ $Cl(X\setminus A)$ = b*ĝ $Cl(A)$ ∩ (X\b*ĝ $Int(A)$) (by result 4.3 (i)) = b*ĝ $Cl(A)$\b*ĝ $Int(A)$ (from (9)) = b*ĝ $Fr(A)$.

(14) b*ĝ $Fr(X\setminus A)$ = b*ĝ $Cl(X\setminus A)$\ b*ĝ $Int(X\setminus A)$ = (X\b*ĝ $Int(A)$)\ (X\b*ĝ $Cl(A)$) (by result 4.3). b*ĝ $Fr(X\setminus A)$ = b*ĝ $Cl(A)$\ b*ĝ $Int(A)$ = b*ĝ $Fr(A)$.

Definition 4.9 If A is a subset of X, then the ***b*ĝ-exterior*** of A is defined by b*ĝ $Ext(A)$ = b*ĝ $(X\setminus A)$.

Theorem 4.10: Let A be a subset of a space X. Then the following statement holds:

1) b*ĝ $Ext(\Phi) = X$;
2) b*ĝ $Ext(X)$ = Φ;
3) $Ext(A)$ ⊆ b*ĝ $Ext(A)$;
4) b*ĝ $Ext(A)$ = X\b*ĝ $Cl(A)$;
5) A is b*ĝ-closed iff b*ĝ $Ext(A) = X\setminus A$;
6) If $A⊆B$, then b*ĝ $Ext(A)$ ⊇ b*ĝ $Ext(B)$;
7) b*ĝ $Ext(A∪B)$ ⊆ b*ĝ $Ext(A)$ ∩ b*ĝ $Ext(B)$;
8) b*ĝ $Ext(A∩B)$ ⊇ b*ĝ $Ext(A)$ ∪ b*ĝ $Ext(B)$;
9) b*ĝ $Ext(A)$ is b*ĝ-open;
10) b*ĝExt(X\b*ĝ$Ext(A)$)=b*ĝ$Ext(A)$;
11) b*ĝ Ext(b*ĝ $Ext(A)$) = b*ĝ Int(b*ĝ $Cl(A)$);
12) b*ĝ $Int(A)$ ⊆ b*ĝ Ext(b*ĝ $Ext(A)$);
13) X = b*ĝ $Int(A)$ ∪ b*ĝ (A) ∪ b*ĝ $Fr(A)$.

Proof: (1) b*ĝ (Φ) = b*ĝ $Int(X\setminus\Phi)$ = b*ĝ $Int(X)$ = X (by theorem 3.8 (2)).

(2) b*ĝ $Ext(X)$ = b*ĝ $Int(X\setminus X)$ = b*ĝ $Int(\Phi)$ = Φ (by theorem 3.8 (1)).

(3) Let $x \in Ext(A)$. Then by definition 2.2 (3), $x \in Int(X\setminus A)$. By theorem 3.7, x ∈ b*ĝ $Int(X\setminus A)$ = b*ĝ $Ext(A)$. Hence, $Ext(A)$ ⊆ b*ĝ $Ext(A)$.

(4) Let x ∈ b*ĝ $Ext(A)$ ⇔ x ∈ b*ĝ $(X\setminus A)$ ⇔ x ∈ X\b*ĝ $Cl(A)$ (by result 4.3 (ii)). Hence, b*ĝ$Ext(A)$ = X\b*ĝ $Cl(A)$.

(5) Let A be b*ĝ-closed. Then $X\setminus A$ is b*ĝ-open. By remark 3.6, b*ĝ $Int(X\setminus A)$ = $X\setminus A$. This implies that b*ĝ $Ext(A)$ = $X\setminus A$. Conversely, let b*ĝ $Ext(A)$ = $X\setminus A$. Then b*ĝ $Int(X\setminus A)$ = $X\setminus A$. Again by remark 3.6, $X\setminus A$ is b*ĝ-open. Hence, A is b*ĝ-closed.

(6) b*ĝ$Ext(A)$ = b*ĝ $Int(X\setminus A)$ = X\b*ĝ $Cl(A)$ (by result 4.3)
⊇ X\b*ĝ(B) (since A ⊆ B and by theorem 3.11(4))
= b*ĝ $Int(X\setminus B)$ = b*ĝ (B) (by definition 4.9).
Hence, b*ĝ $Ext(A)$ ⊇ b*ĝ (B).

(7) Since A ⊆ $A∪B$ and by (6), b*ĝ $Ext(A∪B)$ ⊆ b*ĝ (A). Similarly since B ⊆ $A∪B$ and by (6), b*ĝ $Ext(A∪B)$ ⊆ b*ĝ (B). Hence, b*ĝ $Ext(A∪B)$ ⊆ b*ĝ $Ext(A)$ ∩ b*ĝ $Ext(B)$.

(8) Since A∩B ⊆ A and by (6), b*ĝ $Ext(A)$ ⊆ b*ĝ $(A∩B)$. Similarly since A ∩ B ⊆ B and by (6), b*ĝ $Ext(B)$ ⊆ b*ĝ (A ∩ B). Hence, b*ĝ $Ext(A)$ ∪ b*ĝ (B) ⊆ b*ĝ $Ext(A∩B)$.

(9) follows from definition 4.9 and theorem 3.8(2).

(10) b*ĝ Ext(X\b*ĝ $Ext(A)$) = b*ĝ Ext(X \ b*ĝ$Int(X\setminus A)$) = b*ĝInt(X \ {X \ b*ĝ$Int(X\setminus A)$}) = b*ĝ Int(b*ĝ $Int(X\setminus A)$) =b*ĝ $Int(X\setminus A)$ (by theorem 3.8 (9)) which is b*ĝ $Ext(A)$. Hence, b*ĝ (X\b*ĝ $Ext(A)$) = b*ĝ $Ext(A)$.

(11) b*ĝExt(b*ĝ$Ext(A)$) = b*ĝInt(X \ b*ĝ$Ext(A)$) = b*ĝInt(X\b*ĝ$Int(X\setminus A)$) = b*ĝInt (b*ĝCl ($X\setminus(X\setminus A)$) = b*ĝInt(b*ĝ $Cl(A)$) (by result 4.3 (i)). Hence, b*ĝ Ext(b*ĝ $Ext(A)$) = b*ĝ Int(b*ĝ $Cl(A)$).

(12) Since A ⊆ b*ĝ(A), b*ĝ$Int(A)$ ⊆ b*ĝInt(b*ĝ$Cl(A)$) = b*ĝExt(b*ĝ$Ext(A)$) (from (11)).

(13) b*ĝ $Int(A)$ ∪ b*ĝ $Ext(A)$ ∪ b*ĝ $Fr(A)$ = b*ĝ $Int(A)$ ∪ b*ĝ $Int(X\setminus A)$ ∪ b*ĝ $Fr(A)$ = X (from theorem 4.8 (12)).

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