Separation Axioms via Regular *- Open Set

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Abstract- **The aim of this paper is to introduce new separation axioms regular*-regular and r*-regular using regular*-open sets and investigate their properties. We also study the relationships among themselves and with known axioms regular, normal, semi-regular and semi-normal.**

KEYWARDS: Regular*-regular, r*-regular

Mathematical Subject Classification: 54D10, 54D15.

INTRODUCTION

Separation axioms are useful in classifying topological spaces. Maheswari and Prasad introduce the notion of s-regular and s-normal spaces using semi-open sets. Dorsett introduce the concept of semi-regular and semi-normal spaces and investigated their properties.

In this paper, we define regular*-regular, regular*-normal, r*-regular and r*-normal using regular*-open sets and investigate their properties. We further study the relationships among themselves and with known axioms regular, normal, semi-regular and semi-normal.

PRELIMINARIES:

Throughout this paper (X,τ) will always denote topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X,τ), *Cl*(A) and *Int*(A) respectively denote the closure and the interior of A in X.

Definition 2.1: [7] A subset A of a topological space (X, τ) is called (i) **generalized closed** (briefly g-closed) if *Cl(A)⊆U* whenever A⊆U and U is open.
(ii) **generalized open** (briefly g-open) if X\A is g-closed in X.

Definition 2.2: [6] Let A be a subset of X. Then
(i) **generalized closure** of A is defined as the intersection of all g-closed sets containing A and is denoted by *Cl*(A).*
(ii) **generalized interior** of A is defined as the union of all g-open subsets of A and is denoted by *Int*(A).*

Definition 2.3: [13] A subset A of a topological space (X,τ) is (i) **Regular*-open** (resp. pre-open, regular-open, semi-open) if *A=Int(Cl*(A)) (resp. A⊆Int(Cl(A)), A=Int(Cl(A)), A⊆Cl(Int(A))).*
(ii) **Reguler*-closed** (resp. pre-closed, regular-closed, semi-closed) if *A=Cl(Int*(A)) (resp. Cl(Int(A))⊆A, A=Cl(Int(A)), Int(Cl(A))⊆A).*

The class of all regular*-open (resp. regular*-closed) sets is denoted by R*O(X,τ) (resp. R*C(X,τ)).

Definition 2.4: Let A be a subset of X. Then
(i) the **regular*-closure** of A is defined as the intersection of all regular*-closed sets containing A and is denoted by *r*Cl(A).*
(ii) the **regular*-interior** of A is defined as the union of all regular*-open sets of X contained and is denoted by *r*Int(A).*

Theorem 2.5: Let A⊆X and let x∈X and *r*Cl(A)* is regular*-closed. Then x∈*r*Cl(A)* if and only if every regular*-open set in X containing x intersects A.

Theorem 2.6: (i) Every regular*-open set is open.
(ii) Every regular*-open set is pre-open.
(iii) Every regular*-closed set is closed.

Definition 2.7: A space X is said to be T_1 if for every pair of distinct points x and y in X, there is an open set U containing x but not y and an open set V containing y but not x.

Definition 2.8: A space X is R_0 if every open set contains the closure of each of its points.

Theorem 2.9: (i) X is R_0 if and only if for every closed set F, $Cl(\{x\})\cap F=\phi$, for all x∈X\F.

Definition 2.10: A topological space X is said to be
(i) regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint open sets U and V in X containing x and B respectively.
(ii) s-regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint semi-open sets U and V in X containing x and B respectively.

(iii) semi-regular if for every pair consisting of a point x and a semi-closed set B not containing x, there are disjoint semi-open sets U and V in X containing x and B respectively.

Definition 2.11: A topological space X is said to be

(i) normal if for every pair of disjoint closed sets A and B in X, there are disjoint open sets U and V in X containing A and B respectively.

(ii) s-normal if for every pair of disjoint closed sets A and B in X, there are disjoint semi-open sets U and V in X containing A and B respectively.

(iii) semi-regular if for every pair of disjoint semi-closed sets A and B in X, there are disjoint semi-open sets U and V in X containing A and B respectively.

Definition 2.12: A function $f{:}X\rightarrow Y$ is said to be

(i) closed if $f(V)$ is closed in Y for every closed set V in X.

(ii) regular*-continuous if $f^{-1}(V)$ is regular*-open in X for every open set V in Y.

(iii) regular*-irresolute if $f^{-1}(V)$ is regular*-open in X for every regular*-open set V in Y.

(iv) contra-regular*-irresolute if $f^{-1}(V)$ is regular*-closed in X for every regular*-open set V in Y.

(v) regular*-open if $f(V)$ is regular*-open in Y for every open set V in X.

(vi) pre-regular*-open if $f(V)$ is regular*-open in Y for every regular*-open set V in X.

(vii) contra-pre-regular*-open if $f(V)$ is regular*-closed in Y for every regular*-open set V in X.

(viii) pre-regular*-closed if $f(V)$ is regular*-closed in Y for every regular*-closed set V in X.

Lemma 2.13: If A and B are subsets of X such that $A\cap B = \phi$ and A is regular*-open in X, then $A\cap r^*Cl(B) = \phi$.

Theorem 2.14: A function $f{:}X\rightarrow Y$ is regular*-irresolute if $f^{-1}(F)$ is regular*-closed in X for every regular*-closed set F in Y.

REGULAR SPACES ASSOCIATED WITH REGULAR*-OPEN SETS.

In this section we introduce the concepts of regular*-regular and r*-regular spaces. Also we investigate their basic properties and study their relationship with already existing concepts.

Definition 3.1: A space X is said to be regular*-regular if for every pair consisting of a point x and a regular*-closed set B not containing x, there are disjoint regular*-open sets U and V in X containing x and B respectively.

Theorem 3.2: In a topological space X, the following are equivalent:

(i) X is regular*-regular.

(ii) For every $x\in X$ and every regular*-open set U containing x, there exists a regular*-open set V containing x such that $r^*Cl(V)\subseteq U$.

(iii) For every set A and a regular*-open set B such that $A\cap B \neq \phi$ there exists a regular*-open set U such that $A\cap U \neq \phi$ and $r^*Cl(U)\subseteq B$.

(iv) For every non-empty set A and regular*-closed set B such that $A\cap B = \phi$, there exist disjoint open sets U and V such that $A\cap U \neq \phi$ and $B\subseteq V$.

Proof: (i)⇒(ii): Let U be a regular*-open set containing x, then $B = X\backslash U$ is a regular*-closed set not containing x. Since X is regular*-regular, there exists disjoint regular*-open sets V and W containing x and B respectively. If $y\in B$, W is a regular*-open set containing y that does not intersects V and hence by theorem 2.5, y cannot belong to $r^*Cl(V)$. Therefore $r^*Cl(V)$ is disjoint from B. Hence $r^*Cl(V)\subseteq U$.

(ii)⇒(iii): Let $A\cap B \neq \phi$ and B be regular*-open, let $x\in A\cap B$. Then by assumption, there exists a regular*-open set U containing x such that $r^*Cl(U)\subseteq B$. Since $x\in A$, $A\cap U \neq \phi$. This proves (iii).

(iii)⇒(iv): Suppose $A\cap B = \phi$, where A is non-empty and B is regular*-closed, then $X\backslash B$ is regular*-open and $A\cap(X\backslash B) \neq \phi$. By (iii), there exists a regular*-open set U such that $A\cap U \neq \phi$ and $U \subseteq r^*Cl(U) \subseteq X\backslash B$. Put $V = X\backslash r^*Cl(U)$. Hence V is regular*-open set containing B such that $U\cap V = U\cap(X\backslash r^*Cl(U)) \subseteq U\cap(X\backslash U) = \phi$. This proves (iv).

(iv)⇒(i): Let B be regular*-closed and $x\notin B$. Take $A = \{x\}$, then $A\cap B = \phi$. By (iv), there exist disjoint regular*-open sets U and V such that $U\cap A \neq \phi$ and $B\subseteq V$. Since $U\cap A \neq \phi$, $x\in U$, this proves that X is regular*-regular.

Theorem 3.3: Let X be a regular*-regular space. Then

(i) Every regular*-open set in X is a union of regular*-closed sets.

(ii) Every regular*-closed set in X is an intersection of regular*-open sets.

Proof: (i) Suppose X is regular*-regular. Let G be a regular*-open set and $x\in G$, then $F = X\backslash G$ is regular*-closed and $x\notin F$. Since X is regular*regular, there exist disjoint regular*-open sets U_x and V in X such that $x\in U_x$ and $F\subseteq V$. Since $U_x\cap F\subseteq U_x\cap V = \phi$, we have $U_x\subseteq X\backslash F = G$. Take $V_x = r^*Cl(U_x)$ and V_x is regular*-closed, then by Lemma 2.13, $V_x\cap V = \phi$. Now $F\subseteq V$ implies that $V_x\cap F \subseteq V_x\cap V = \phi$. It follows that $x\in V_x \subseteq X\backslash F = G$. This proves that $G = \cup\{ V_x : x \in G\}$. Thus G is a union of regular*-closed sets.

(ii) Follows from (i) and set theoretic properties.

Theorem 3.4: If f is a regular*-irresolute and pre-regular*-closed injection of a topological space X into a regular*-regular space Y, then X is regular*-regular.

Proof: Let $x \in X$ and A be a regular*-closed set in X not containing x. Since f is pre-regular*-closed, $f(A)$ is regular*-closed set in Y not containing $f(x)$. Since Y is regular*-regular, there exist disjoint regular*-open sets V_1 and V_2 in Y such that $f(x)\in V_1$ and $f(A)\subseteq V_2$. Since f is regular*-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint regular*-open sets in X containing x and A respectively. Hence X is regular*-regular.

Theorem 3.5: If f is a regular*-continuous and closed injection of a topological space X into a regular space Y then X is regular*-regular.
Proof: Let x ∈ X and A be a regular*-closed set in X not containing x, then by Theorem 2.6 , A is closed in X. Since f is closed, $f(A)$ is closed set in Y not containing $f(x)$. Since Y is regular, there exist disjoint open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $f(A) \subseteq V_2$. Since f is regular*-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint regular*-open sets in X containing x and A respectively. Hence X is regular*-regular.

Theorem 3.6: If $f: X \to Y$ is a regular*-irresolute bijection which is pre-regular*-open and X is regular*-regular, then Y is also regular*-regular.
Proof: Let $f: X \to Y$ is a regular*-irresolute bijection which is pre-regular*-open and X is regular*-regular. Let y ∈ Y and B be a regular*-closed set in Y not containing y. Since f is regular*-irresolute, by Theorem 2.14, $f^{-1}(B)$ is regular*-closed set in X not containing $f^{-1}(y)$. Since X is regular*-regular, there exist disjoint regular*-open sets U_1 and U_2 in X containing $f^{-1}(y)$ and $f^{-1}(B)$ respectively. Since f is pre-regular*-open, $f(U_1)$ and $f(U_2)$ are disjoint regular*-open sets in Y containing y and B respectively. Hence Y is regular*-regular.

Theorem 3.7: If f is a continuous regular*-open bijection of a regular space X into a space Y then Y is regular*-regular.
Proof: Let y ∈ Y and B be a regular*-closed set in Y not containing y, by Theorem 2.6, B is closed in Y. Since f is continuous bijection $f^{-1}(B)$ is a closed set in X not containing the point $f^{-1}(y)$. Since X is regular, there exist disjoint open sets U_1 and U_2 in X containing $f^{-1}(y)$ and $f^{-1}(B)$ respectively. Since f is regular*-open, $f(U_1)$ and $f(U_2)$ are disjoint regular*-open sets in Y containing y and B respectively. Hence Y is regular*-regular.

Theorem 3.8: If X is regular*-regular, then it is regular*-R_0.
Proof: Suppose X is regular*-regular. Let U be a regular*-open set and x ∈ U. Take F = X\U, then F is regular*-closed set not containing x. By regular*-regularity of X, there are disjoint regular*-open sets V and W such that x ∈ V and F⊆W. If y ∈ F, then W is regular*-open set containing y that does not intersect V. Therefore y ∉ $r^*Cl(V)$ which implies y ∉ $r^*Cl(\{x\})$. That is $r^*Cl(\{x\}) \cap F = \phi$ and hence $r^*Cl(\{x\}) \subseteq X \setminus F = U$. Hence X is regular*-$R_0$.

Definition 3.9: A space X is said to be r*-regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint regular*-open sets U and V in X containing x and B respectively

Theorem 3.10: (i) Every r*-regular space is regular.
(ii) Every r*-regular space is s-regular.
Proof: (i) Suppose X is r*-regular. Let F be a closed set and x ∉ F. Since X is r*-regular,
there exist disjoint regular*-open sets U and V containing x and F respectively. By Theorem 2.6, U and V are open in X. This implies that X is regular.
(ii). Follows from (i) and the fact that every open set is semi-open.

Theorem 3.11: For a topological space X, the following are equivalent:
(i) X is r*-regular.
(ii) For every x ∈ X and every open set U containing x, there exists a regular*-open set V containing x such that $r^*Cl(V) \subseteq U$.
(iii) For every set A and an open set B such that A∩B = ϕ, there exists a regular*-open set U such that A∩U ≠ ϕ and $r^*Cl(U) \subseteq B$.
(iv) For every non-empty set A and closed set B such that A∩B = ϕ, there exist disjoint regular*-open sets U and V such that A∩U ≠ ϕ and B ⊆ V.
Proof: (i)⇒(ii): Let U be an open set containing x, then B = **X\U** is closed set not containing x. Since X is r*-regular, there exist disjoint regular*-open sets V and W containing x and B respectively. If y ∈ B, W is a regular*-open set containing y that does not intersects V and hence by Theorem 2.5, y cannot belong to $r^*Cl(V)$. Therefore $r^*Cl(V)$ is disjoint from B. Hence $r^*Cl(V) \subseteq U$.
(ii)⇒(iii): Let A∩B ≠ ϕ and B be open. Let x ∈ A∩B, then by assumption, there exists a regular*-open set U containing x such that $r^*Cl(U) \subseteq B$. Since x ∈ A, A∩U ≠ ϕ. This proves (iii).
(iii)⇒(iv): Suppose A∩B = ϕ, where A is non-empty and B is closed. Then X\B is open and A∩(X\B) ≠ ϕ. By (iii), there exists a regular*-open set U such that A∩U ≠ ϕ and U ⊆ $r^*Cl(U) \subseteq X \setminus B$. Put V = X\$r^*Cl(U)$ and take $r^*Cl(U)$ is regular*-closed. Hence V is a regular*-open set containing B such that U∩V = U∩(X\$r^*Cl(U)$) ⊆ U∩(X\U) = ϕ. This proves (iv).
(iv)⇒(i): Let B be closed and x ∉ B. Take A = {x}, then A∩B = ϕ. By (iv), there exist disjoint regular*-open sets U and V such that U∩A ≠ ϕ and B ⊆ V. Since U∩A ≠ ϕ, x ∈ U. This proves that X is r*-regular.

Theorem 3.12: Every regular*-regular space is r*-regular.
Proof: Suppose X is regular*-regular. Let F be a regular*-closed set and x ∉ F, then by Theorem 2.6, F is closed in X. Since X is regular*-regular, there exist disjoint regular*-open sets U and V containing x and F respectively. This implies that X is r*-regular.

Theorem 3.13: (i) Every r*-regular T_1 space is regular*-T_2.
(ii) Every regular*-regular regular*-T_1 space is regular*-T_2.
Proof: (i) Suppose X is r*-regular and T_1. Let x and y be two disjoint point in X. Since X is T_1, {x} is closed and y ∉ {x}. Since X is r*-regular, there exist disjoint regular*-open sets U and V in X containing {x} and y respectively. It follows that X is regular*-T_2.
(ii). Suppose X is regular*-regular and regular*-T_1. Let x and y be two distinct points in X. Since X is regular*-T_1, {x} is regular*-closed and y ∉ {x}. Since X is regular*-regular, there exist disjoint regular*-open sets U and V in X containing {x} and y respectively. It follows that X is regular*-T_2.

Theorem 3.14: Let X be a r*-regular space.

(i) Every open set in X is a union of regular*-closed sets.

(ii) Every closed set in X is an intersection of regular*-open sets.

Proof: (i) Suppose X is r*-regular. Let G be an open set and $x \in G$, then $F = X\backslash G$ is closed and $x \notin F$. Since X is r*-regular, there exist disjoint regular*-open sets U_x and U in X such that $x \in U_x$ and $F \subseteq U$. Since $U_x \cap F \subseteq U_x \cap U = \phi$, we have $U_x \subseteq X\backslash F = G$. Take $V_x = r^*Cl(U_x)$ and V_x is regular*-closed. Now $F \subseteq U$ implies that $V_x \cap F \subseteq V_x \cap F = \phi$. It follows that $x \in V_x \subseteq X\backslash F = G$. This proves that $G = \cup\{ V_x : x \in G \}$. Thus G is a union of regular*-closed sets.

(ii). Follows from (i) and set theoretic properties.

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