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Total Domination in Graphs – Minus (Signed)

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Abstract: - A function f : V(G) {+1,0,-1} defined on the vertices of a graph G is said to be a minus total dominating function if the sum of the values of its function is atleast one over any open neighborhood. The minus total domination number γ_t^-(G) of G is the minimum weight of a minus total dominating function on G. By simply changing "{+1, 0, -1}" in the above definition to "{+1, -1}", we can define signed total dominating function and the signed total domination number γ_t^s(G) of G. Here we present a sharp lower bound on the signed total domination number for a k-partite graph, which results in a short proof of a result due to Kang et. al. A sharp bounds on 〖γ〗_t^s and γt- for triangle - free graphs are given.

Keywords: Minus total domination; signed total domination; k-partite graph; Triangle free graph; Hypergraph;

INTRODUCTION

Let S be **vertex set** and {A_1, A_2, …,} be the **edge set** of a hypergraph H. Z is a set of integers. P is an arbitrary subset of Z and α is an integer. Let the function f:S→ P defines an α-dominating partition of the hypergraph H with respect to P, if f(A):= $\sum_{x\in A} f(x) \geq \alpha$, for every edge A in H.

The minimum of such functions is defined as the **α-domination number of H** with respect to P:dom_α (H):=min {f(S) : f is α-dominating partition}. In particular, when P ={+1, -1} or {+1, 0, -1}, we get the **signed α-domination number** and **minus α-domination number**, respectively denoted by **m dom**$_\alpha$ and **s dom**$_\alpha$.

Here now we consider a **simple graph** G=(V, E), with **vertex set** V and **edge set** E. Let v be a vertex in V. The open neighborhood of v, $N_G(v)$, is defined as the set of vertices adjacent to v, ie $N_G(v)$={u| uv∈E}. The **closed neighborhod** of v is $N_G[v]=N_G(v)\cup\{v\}$ when $S \subseteq V(G)$, denote by G[s], the graph induced by S. If A, B contained in V(G) (ie A, B ⊆ V(G)), A∩B = ∅, we denote the number of edges between A and B by e(A, B). Again the **degree of v** in G is denoted by $d_G(v)$, and the **maximum degree** and **minimum degree** of G are respectively denoted by Δ(G) and δ(G). Let k ≥ 2 be an integer.

A Graph G = (V, E) is said to be **k-partite** if V admits a partition into k classes in such a way that every edge has its ends in different classes. Vertices in the same partition class must not be adjacent. Instead of '2 = partite' it is called as **bipartite**. A **triangle – free graph** is a graph containing no cycles of length three.

A **signed total dominating function** of a graph G is normally defined as a function f:V(G) → {+1, -1} such a way that for every vertex v, $\sum_{u\in N(v)} f(u)\geq 1$, and the minimum cardinality of the sum $\sum_{v\in V} f(v)$ over all such functions is said to be a **signed total domination number**, and is denoted as γ_t^s (G), that is

$\gamma_t^s(G)$ = min {f(V(G)): f is a signed dominating function of G}.

A minus total dominating function is also defined as a function of the form f: V→ {+1, 0, -1} such that $\sum_{u\in N(v)} f(u) \geq 1$ for all v∈V. The **minus total domination number** for a graph G is $\gamma_t^-(G)$ = min {f(V(G)):f is a minus total dominating function of G}.

From the definitions above, every signed total dominating function of G is clearly a minus total dominating function of G, hence $\gamma_t^-(G) \leq \gamma_t^s$ (G). Using the notation used in hypergraphs, we see that $\gamma_t^s(G)$ = $s\mathrm{dom}_1(N(G))$ and γ_t^- (G) = mdom$_1(\mathcal{N}(G))$, where $\mathcal{N}$ is the neighborhood hypergraph on the vertex set V(G) and its edges are the open neighborhoods {$N_G(v)$: v∈V(G)}.

It is already shown by experts that the decision problems for the signed and minus total domination numbers of graph are NP-complete respectively, even when the graph is restricted to a bipartite graph or a chordal graph. Many bounds on γ_t^s of graphs were established. Some authors have got sharp upper bounds on γ_t^- for small degree regular graphs.

Here we first give a sharp lower bound on γ_t^s (G) of a k-partite graph G in terms of its order and minimum degree.

2. RESULTS

Theorem 1

Let G = (V,E) be a k-partite graph of order n with $\delta(G)\geq 1$ and let $c = \lceil \delta(G)+1)/2 \rceil$.

Then $\gamma_t^s(G) \geq \frac{K}{K-1}$

$$\left(-(c-1) + \sqrt{(c-1)^2 + 4\,\frac{k-1}{k}\,cn} \right) - n$$

And this bound is sharp.

Proof

Let G = (V, E) be a k-partite graph of order n with vertex classes $V_1, V_2, \ldots, V_k$ and there is no isolated vertex. When n = 2, 3 the result is trivial, hence we suppose that $n \geq 4$. Let $f : V \rightarrow \{+1, -1\}$ be a signed total dominating function on G with $f(V(G)) = \gamma_t^s(G)$ and let P and M be the sets of vertices in V that are assigned the value +1 and -1, respectively under f. Further let $P_i = P \cap V_i$, for $i=1,2,\ldots,k$. Then, $n = |P| + |M|$ and $P = \bigcup_{i=1}^{k} P_i$. We also assume that:

let $|P| = p, |M| = m, |P_i| = p_i$ and $\delta(G)=\delta$. For every vertex $v \in M$, v is adjacent to at least $\lfloor d_G(v)/2 \rfloor + 1$ in P since $f(N(v)) \geq 1$,

so $|N_G(v) \cap P| \geq \lfloor \delta/2 \rfloor + 1 = \lceil (\delta+1)/2 \rceil = c$.

Hence

$$e(P,M) = \sum_{v\in M} |N_G(v) \cap P| \geq c(n-p) \qquad (2.1)$$

on the other hand, for every vertex $v \in P_i$,

it gives that $|N_G(v) \cap M| \leq |N_G(v) \cap (P - P_i)| - 1 \leq p - p_i - 1$

Thus,

$$e(P,M) = \sum_{v\in P} |N_G(v) \cap M| \leq \sum_{i=1}^{k} \sum_{v\in P_i}$$

$$(|N_G(v) \cap (P-P_i)| - 1) \leq \sum_{i=1}^{k} p_i(p - p_i - 1) \qquad (2.2)$$

Also $$k \sum_{i=1}^{k} p_i^2 \geq p^2 \qquad (2.3)$$

Thus, combining with inequalities (2.1) and (2.2) we have

$$c(n-p) \leq e(P,M) \leq \frac{K-1}{K} p^2 - p, \qquad (2.4)$$

reduces to

$$\frac{K-1}{K} p^2 + (c-1)p - cn \geq 0\text{'}$$

Hence,

$$p \geq \left[-(c-1) + \sqrt{(c-1)^2 + 4\,\frac{k-1}{k}\,cn} \right] \Big/ 2\left(\frac{k-1}{k}\right)$$

Therefore

$$\gamma_t^s(G) = 2p - n \geq \frac{k}{k-1} \left[-(c-1) + \sqrt{(c-1)^2 + 4\,\frac{k-1}{k}\,cn} \right] - n.$$

To show that the bound is sharp, we proceed as follows:

For integers $k \geq 2$, let H_i be a complete bipartite graph with vertex classes V_i and U_i, where $|V_i| = k$ and $|U_i| = (k^2 - k - 1)$, for $i=1, 2, \ldots, k$.

We now let H(k) to be the graph obtained from the disjoint union of $H_1, H_2, \ldots, H_k$ by joining each vertex of V_i in H_i with all the vertices of $\bigcup_{j=1,j\neq i}^{k} V_j$, and adding $(k-1)(k^2-k-1)$ edges between U_i with $\bigcup_{j=1,j\neq i}^{k} U_j$ so that each vertex of U_i has exactly $(k-1)$ neighbors in $\bigcup_{j=1,j\neq i}^{k} U_j$ while each vertex of $\bigcup_{j=1,j\neq i}^{k} U_j$ has exactly one neighbor in U_i for all i=1, 2, …, k. Let $Y_i = V_i \cup U_{i+1}$, where $i+1 \pmod k$. Then H(k) is a k-partite graph of order $n = k(k^2-1)$ with vertex classes $Y_1, Y_2, \ldots, Y_k$ and $|Y_i| = k^2 - 1$. The graph H(3) is given in Fig 1. Note that each vertex of U_i has a minimum degree 2k-1. Assigning to each vertex of $\bigcup_{i=1}^{k} V_i$ the value +1 and to each vertex of $\bigcup_{i=1}^{k} U_i$ the value -1, we produce a total signed dominating function f of H with weight equal to

$$f(V(H(k))) = k^2 - k(k^2 - k - 1)$$

$$= k(-k^2 + 2k + 1)$$

$$= \frac{k}{k-1}\left(-(c-1) + \sqrt{(c-1)^2 + 4\,\frac{k-1}{k}\,cn} \right) - n$$

Consequently,

$$\gamma_t^s(H(k)) = \frac{k}{k-1} - (c-1) + \sqrt{(c-1)^2 + 4\,\frac{k}{k-1}\,cn} - n$$

It is already shown that for a bipartite graph G,

$$\gamma_t^s(G) \geq 2\sqrt{2n} - n.$$

By our Theorem 1, we can easily extend the result to k-partite graphs and characterize the external graphs achieving this bound. For this act, we introduce a family τ of graphs as follows:

For integer $r \geq 1$, $k \geq 2$, let H_i (i=1,2,…,k) be the graph obtained from the disjoint union of r stars $K_{1,(k-1)r-1}$ (the graph $K_{1,0}$ is regarded as K_1 when $r = 1$ and $k = 2$) with centers $V_i = \{x_{ij} \mid j = 1,2,\ldots, r\}$. Apart from this, let U_i denote the set of vertices of degree 1 in H_i that are not central vertices of stars and write $X_i = V_i \cup U_{i+1}$, where i+1 (mod k). We now let $G_{k,r}$ be the k-partite graph obtained from the disjoint union of $H_1, H_2, \ldots, H_k$ by joining each center of H_i (i=1, 2, .., k) with all the centers of $\bigcup_{j=1, j\neq i}^{k} H_j$. By our construction, we know that $G_{k,r}$ is a is a k-partite graph of order $n = k\,(k-1)\,r^2$ with the vertex classes $X_1, X_2, \ldots, X_k$ and $|X_i| = (k-1)r^2$. Let $\tau = \{G_{k,r} \mid r \geq 1, k \geq 2\}$.

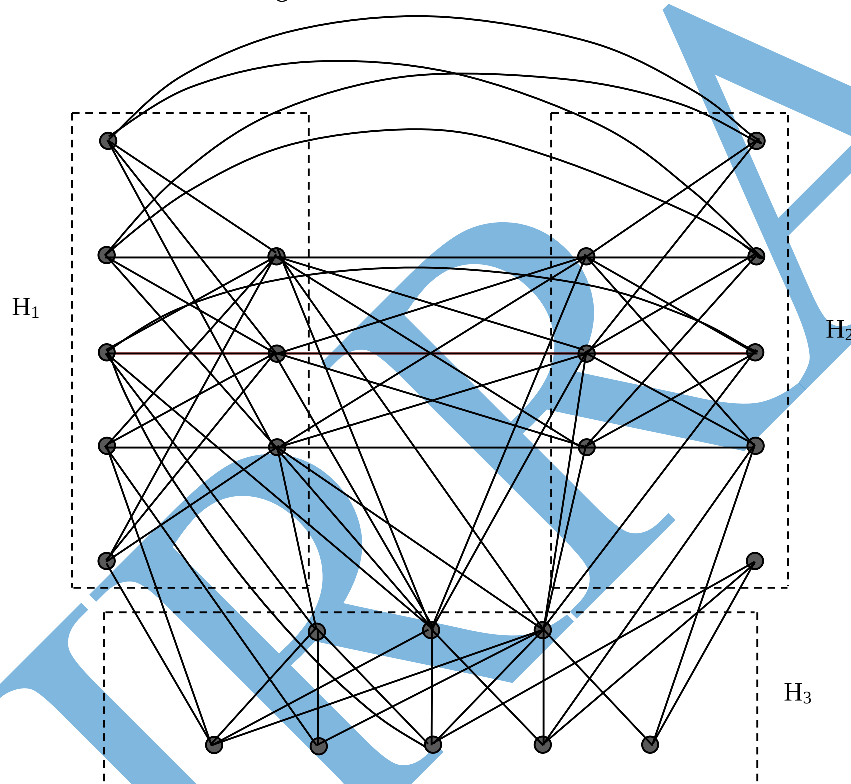

Fig 1 (below) : The H(3) Graph

Theorem 2

If G = (V, E) is a k-partite graph of order n with no isolated vertex, then

$$\gamma_t^s(G) \geq 2\sqrt{\frac{k}{k-1}n} - n\,,$$

were the equality holds if and only if G∈τ.

Proof:

Let $g(x) = \frac{k}{k-1}\left(-x + \sqrt{x^2 + 4\,\frac{k-1}{k}(x+1)n}\right) - n$. It is now easy to check that $g'(x) > 0$ if n ≥ 2, hence g(x) is a strictly monotone increasing function where x ≥ 0. Note that c ≥1, hence $\gamma_t^s(G) \geq g(c-1) \geq g(0)$ which gives the desired bound.

$$\text{If } \gamma_t^s(G) = 2\sqrt{kn/(k-1)} - n,$$

then $c = 1$ since g(x) is a strictly monotone function, and thus $\delta = 1$. Further all the equalities in (2.1), (2.2) and (2.3) hold. The equality now in (2.3) implies that $p_1 = p_2 = \ldots p_k{:} = r$. The equalities in (2.1) and (2.2) imply that each vertex of M has degree 1 and is exactly adjacent to a vertex of P, while each vertex of P_i has degree $p - p_i = kr - r$ in

G[P] and has exactly $p - p_i - 1 = r(k-1) - 1$ neighbors in M. If follows that G$\in\tau$.

On the other hand, if we suppose that G$\in\tau$. Then, there exist integers $r \geq 1$, k ≥ 2 such that G = $G_{k,r}$. Assuming to all kr central vertices of the stars the value +1, and to all other vertices the value -1, we produce a signed total dominating function f of weight f(V(G)) = $kr - kr(2k-1)$ = $2kr - 2k^2r$. = $2\sqrt{kn/(k-1)} - n$

We now present a short-cut proof and also further give a characterization of the extremal graphs.

Theorem 3

If G = (V,E) is a k-partite graph of order n with no isolated vertex, then we have $\gamma_t^-(G) \geq 2\sqrt{\frac{k}{k-1}}n - n$, where the equality holds if and only if G$\in\tau$.

Proof :

Let f: V→{+1, 0, -1} be a minus total dominating function on G with f(V(G)) = γ_t^-(G) and let Q be the set of vertices in V(G) that are assigned the value 0. Further, let G′= G - Q and that G′ is a k′-partite graph of order n′.

Then $2 \leq k' \leq k$ and $2 \leq n' \leq n$

Clearly; $f' = f/G'$ is a signed total dominating function on G′, Hence

γ_t^s (G′) ≤ $f'(V(G')) = f(V(G))$. Now by Theorem 2, we have

$$\gamma_t^-(G) \geq \gamma_t^s(G') \geq 2\sqrt{\frac{k'}{k'-1}}n' - n'.$$

Denote $h(x,y)$ as $2\sqrt{yx/(y-1)} - x$. Now on partial differentiation, we see that $\partial h(x,y)|\,\partial x < 0$ and $\partial h(x,y)/\,\partial y < 0$ for $x, y \geq 2$ hence $h(x,y)$ is a strictly monotone decreasing function on variables x and y, respectively. This indicates that

$$\gamma_t^-(G) \geq \gamma_t^s(G') \geq 2\sqrt{\frac{k}{k-1}}n - n.$$

The theorem below implies the fact that the equality holds if and only if G$\in\tau$.

By theorem 2 and 3, we obtain the following extremal result on the minus total domination and signed total domination of a k-partite graph.

Theorem 4

If G = (V, E) is a k-partite graph of order n with no isolated vertex, then the following statements are equivalent.

i. $\gamma_t^s(G) = 2\sqrt{\frac{k}{k-1}}n - n$;

ii. $\gamma_t^-(G) = 2\sqrt{\frac{k}{k-1}}n - n$;

iii. G ∈ τ

Proof

By theorem 2 and 3, we have

$\gamma_t^s(G) = \gamma_t^-(G) \geq 2\sqrt{kn/(k-1)} - n$, hence it is enough to prove that (ii) ⟹ (iii). We here use the notation used while proving theorem 3 above.

Accordingly: If $\gamma_t^-(G) = 2\sqrt{kn/(k-1)} - n$, then $h(k',n') = h(k,n)$. Also note that $h(x,y)$ is a strictly monotone function on variables x and y respectively, where $x, y \geq 2$.

This implies that $k' = k, n' = n$. Hence $Q = \phi$ and this gives that f is also a minimum signed total dominating function, ie. $\gamma_t^s(G) = 2\sqrt{kn/(k-1)} - n$. Now the result simply follows from Theorem 2.

Now recall a subclass, constructed already, of τ. Clearly, each $G_{2,r}$ of $\mathcal{F}$ is a bipartite graph of order n = $2r^2$ with vertex classes X_1, X_2 and $|X_i| = r^2$. As a special case of Theorem 4, we have

Theorem 5

If G is a bipartite graph of order n with δ(G) ≥1, then $\gamma_t^s(G) \geq 2\sqrt{2n} - n$, the equality holds if and only if G∈ $\mathcal{F}$.
We now have a known and useful result.

Statement

For any triangle-free graph G of order p, |E(G)| ≤ $p^2/4$, the equality is true if and only if G = $K_{\frac{p}{2},\frac{p}{2}}$ and G = $K_{\frac{p}{2},\frac{p}{2}}$ is a balance complete bipartite graph.

Theorem 6

Let G be a triangle – free graph of order n with δ(G) ≥1 and let $c = \lceil (\delta(G)+1)/2 \rceil$ then

$$\gamma_t^s(G) \geq 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n \qquad \text{(a)}$$

Proof

We first prove the inequality (a) for a triangle free graph G. Let $f: V \rightarrow \{+1, -1\}$ be a signed total dominating function of G with $f(V(G)) = \gamma_t^s$ (G) and let $P = \{v \in V(G) | f(v) = +1\}$,

$M = \{v \in V(G) | f(v) = -1\}$. Further, let |P| = p and |M| = m. Obviously, P∪M is a partition of V(G). Then γ_t^s (G) = |P| - |M| = $2p - m$.

Using the argument of Theorem1, by estimating the number of edges between P and M, we get

$$e(P, M) = \sum_{v \in M} |N_G(v) \cap P| \geq cm \quad (2.5)$$

and also $e(P, M) = \sum_{v \in P} |N_G(v) \cap M| \leq$

$$\sum_{v \in P} (|N_G(v) \cap P)|-1) = \sum_{v \in P} d_{G[P]}(v) - p \quad (2.6)$$

We have By a known Lemma:

For any triangle-free graph G of order P, $|E(G)| \leq P^2/4$ where equality holds if and only if G = $K_{\frac{p}{2},\frac{p}{2}}$ and $K_{\frac{p}{2},\frac{p}{2}}$, is a balance complete bipartite graph. Using this, we obtain

$$c(n-p) \leq e(P, M) \leq 2|E(G[P])| - p \leq \frac{p^2}{2} - p \quad (2.7)$$

the above implies that

$p \geq -(c-1) + \sqrt{(c-1)^2 + 2cn}$. Hence,

$$\gamma_t^s(G) = 2p - n \geq 2\left(-(c-1) + \sqrt{(c-1)^2 + 2cn}\right) - n$$

Applying the results in Theorem 6, we obtain the following

Theorem 7

If G is a triangle-free graph of order n with δ(G) ≥1, then

$$\gamma_t^-(G) \geq 2\sqrt{2n} - n,$$

where equality holds if and only if G∈$\mathcal{F}$.

Proof:

Let $h_1(x) = 2\left(-x + \sqrt{x^2 + 2(x+1)n}\right) - n$

It is easy to check that $h_1(x)$ is strictly monotone increasing function when x ≥ 0 and n ≥2. Hence, by Theorem 6, we have γ_t^s (G) ≥ $2\sqrt{2n} - n$

We now show that $\gamma_t^-(G) \geq 2\sqrt{2n} - n$ Let f: V→{+1, 0, -1} be a minus total dominating function on G with f(V(G))= γ_t^-(G) and let Q be the set of vertices in V(G) that are assigned the value 0. Apart from this Let G′ = G - Q and |V(G′)| = n′.Then G′ is triangle – free. Clearly, f' = f/G′ is a signed total dominating function on G′, hence

$$\gamma_t^s(G') \leq f'(V(G')) = f(V(G)).$$

We can also see that $h_2(x) = 2\sqrt{2x} - x$ is a strictly monotonic decreasing function for $x > 1$. Hence

$$\gamma_t^-(G) \geq \gamma_t^s(G') \geq 2\sqrt{2n'} - n' \geq 2\sqrt{2n} - n$$

If $\gamma_t^-(G) = 2\sqrt{2n} - n$, then n′ = n since $h_2(x)$ is a strictly monotonic function. This shows that Q=ϕ. Hence f is a signed total dominating function on G, and thus

$\gamma_t^s(G) \leq \gamma_t^-(G)$, which implies

$$\gamma_t^s(G) = \gamma_t^-(G) = 2\sqrt{2n} - n$$

This gives that γ_t^s (G) = $h_1(0)$;and hence $c = 1$ and the equality holds for the in equalities in the equation, (2.5), (2.6) and (2.7) in the proof of Theorem 6.

The chain of equality in (2.7) implies that $|E(G[P])| = p^2/4$.

By the “Known Lemma” given above, G[P] is a balanced complete bipartite graph $K_{\frac{p}{2},\frac{p}{2}}$. Apart from this, the chain of equalities implies that each vertex of M has degree 1 and is precisely adjacent to a vertex of P, while each vertex of P has degree p −1 and is precisely adjacent to (p/2) −1 vertices of M. Now G is a bipartite graph.

By theorem 4, If G is a bipartite graph of order n with δ(G) ≥1, then γ_t^s (G) ≥ $2\sqrt{2n} - n$, where the equality is true if f G∈$\mathcal{F}$.

Using this we have G∈$\mathcal{F}$ in our case , on the other hand if we suppose G∈$\mathcal{F}$ then by the same result above, γ_t^s (G) = $2\sqrt{2n} - n$.

Since $\gamma_t^s(G) \geq \gamma_t^-(G) \geq 2\sqrt{2n} - n$, we have $\gamma_t^-(G) = 2\sqrt{2n} - n$.

In view of the above results we have

Theorem 8

If G is a triangle – free graph of order n with $\delta(G) \geq 1$, then the following statements are equivalent.

i. $\gamma_t^s(G) = 2\sqrt{2n} - n$,
ii. $\gamma_t^-(G) = 2\sqrt{2n} - n$,
iii. $G \in \mathcal{F}$

3. CONCLUSION

The minus (reps. signed) total domination problem can be taken as a generalization of both the classical total domination problems and the minus domination problems. In the above, illustrated the lower bounds on minus and signed total domination numbers of k-partite graphs and triangle-free graphs and extremal graphs achieving these bounds. The methods used may also be used to characterize the extremal graphs of k-partite graphs attaining the lower bound.

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