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Open and Closed maps via P̂g sets

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Abstract- **The investigation of generalized closed (g- closed) sets in a topological space was initiated by Levine [4] and concept of T½ spaces was introduced. Dunham [3] additionally explored the properties of T½ spaces and defined a new closure operator cl* by using generalized closed sets. S. Pious Missier and S. Jackson [7] cleared another pathway by introducing a new notion of generalized closed sets called P̂g closed sets. This paper explores P̂g open and P̂g closed maps in topological spaces and concentrate some of its essential properties and relations among them. In this paper we derived some important results and establish its relationship with other existing open and closed maps in topological spaces.**

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1. INTRODUCTION

The concept of generalized closed sets introduced by Levine [4] plays a significant role in General Topology. This notion has been studied extensively in recent years by many topologists. The investigation of generalized closed sets has led to several new and interesting concepts. Dunham [3] further investigated the Properties of $T_{1/2}$ spaces and defined a new Closure operator cl* by using generalized Closed sets.

In 1996, H. Maki, J. Umehara and T. Noiri [5] introduced the Class of Pre generalized Closed sets and used them to obtain Properties of Pre-$T_{1/2}$ spaces. The modified forms of generalized closed sets and generalized continuity were studied by K. Balachandran, P. Sundaram and H. Maki [1].

M.K.R.S. Veerakumar et.al [11] introduced a new Classes of Open sets namely g* - closed sets .This characterization paved a new pathway. Dr. S. Pious missier and S. Jackson [7] introduced a new class of generalized closed sets called $\hat{P}g$ closed sets.

This paper is devoted to $\hat{P}g$ open map and $\hat{P}g$ closed map. We establish some of their properties and we obtain the connection between the above maps with some existing maps, to substantiate it suitable examples are given at the respective places.

Throughout this paper, spaces (X, τ), (Y, σ), (Z, η) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The Closure of A and the interior of A are denoted by cl (A) and int (A) respectively. The topological spaces (X, τ), (Y, σ), (Z, η) are represented by X, Y, Z respectively.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space (X, τ) is called

(1) a **Pre-Open set [5]** if A ⊆ int (cl (A)) and a **Pre-Closed** set if cl (int (A)) ⊆ A.

(2) a **g closed set [4]** if cl (A) ⊆ U whenever A ⊆ U , U is Open in (X, τ). The complement of g closed set is **g open**.

(3) a **Pre*-Open set [10]** if A ⊆ int* (cl (A)) and a **Pre*-Closed** set if cl* (int (A)) ⊆ A

(4) a $\hat{P}g$ **closed set [7]** Pre* cl (A) ⊆ U whenever A ⊆ U , U is Pre*-Open in (X, τ).

Definition 2.2: [7] The intersection of all P̂g Closed sets containing A is called **P̂g Closure** of A and denoted by P̂g cl (A). That is P̂g cl (A) = ∩ {F: A ⊆ F and F ∈ P̂g C (X)}

Definition 2.3: [7] Let A be a subset of X. Then **P̂g interior** of A is defined as the union of all P̂g Open sets contained in it. P̂g int (A) = ∪{V: V ⊆ A and V ∈ P̂g O (X)}.

Definition 2.4: [8] A function f: (X, τ) → (Y, σ) is called ***g* Continuous** if f^{-1} (V) is *g* open in (X, τ) for every open set in (Y, σ). i.e.) the pre image of every open set is *g* open.

Definition 2.5: [8] A function f: (X, τ) → (Y, σ) is called $\hat{P}g$ **Continuous** if f^{-1} (V) is $\hat{P}g$ open in (X, τ) for every open set in (Y, σ). i.e.) the pre image of every open set is $\hat{P}g$ open.

Definition 2.6: [8] A function f: (X, τ) → (Y, σ) is called **strongly $\hat{P}g$ Continuous** if f^{-1} (V) is open in (X, τ) for every $\hat{P}g$ open set in (Y, σ). i.e.) the pre image of every $\hat{P}g$ open set is open.

Definition 2.7: [9] A function f: (X, τ) → (Y, σ) is called $\hat{P}g$ **irresolute** if f^{-1} (V) is $\hat{P}g$ open in (X, τ) for every $\hat{P}g$ open set in (Y, σ). i.e.) the pre image of every $\hat{P}g$ open set is $\hat{P}g$ open.

Definition 2.8: [3] A topological space (X, τ) is called a **$T_{1/2}$ space** if every g-closed set in X is closed.

3. $\hat{P}$g Closed map

Definition 3.1: A map f: (X, τ) → (Y, σ) is called a $\hat{P}g$ **closed** if the image of each closed set in X is $\hat{P}g$ closed in Y.

Theorem 3.2: A map f: (X, τ) → (Y, σ) is $\hat{P}$g closed if and only if $\hat{P}$g cl (f (A)) ⊆ f (cl (A)) for each set A in X.

Proof:

Necessity: Suppose that f is $\hat{P}$g closed map. Since for each set A in X, cl (A) is closed set in X, then f (cl (A)) is a $\hat{P}$g closed set in Y. Since, f (A) ⊆ f (cl (A)), then $\hat{P}$g cl (f (A)) ⊆ f (cl (A)).

Sufficiency: Suppose A is a closed set in X. Since $\hat{P}g$ cl (f (A) is the smallest $\hat{P}g$ closed set containing f (A), then f (A) ⊆ $\hat{P}g$ cl (f (A) ⊆ f (cl (A)) = f (A). Thus, f (A) = $\hat{P}g$ cl (f (A)). Hence, f (A) is $\hat{P}g$ closed set in Y. Therefore, f is a $\hat{P}g$ closed map.

Theorem 3.3: If f: (X, τ) → (Y, σ) is a closed map and g: (Y, σ) → (Z, η) is $\hat{P}g$ closed, then the composition g o f: X → Z is $\hat{P}g$ closed map.

Proof: Let O be any closed set in X. Since f is closed map. f (O) is closed set in Y. Since g is $\hat{P}g$ closed map, g (f (O)) is $\hat{P}g$ closed in Z which implies g ∘ f ({O}) = g (f {O}) is $\hat{P}g$ closed and hence, g ∘ f is $\hat{P}g$ closed.

Remark 3.4: If f: (X, τ) → (Y, σ) is $\hat{P}g$ closed map and g: (Y, σ) → (Z, η) is closed map then the composition g o f: X → Z is need not be a $\hat{P}g$ closed map. It can be observed from the following example.

Example 3.5: Consider X = Y = Z = {a, b, c} with topologies τ = {∅, a, b, ab, bc, X}, σ = {∅, ab, X} and η = {∅, a, ab, X}. Define f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be the identity mappings. Then f is a $\hat{P}g$ closed map and g is a closed map but their composition is not a $\hat{P}g$ closed map.

Remark 3.6: The composition of two $\hat{P}g$ closed maps need not be a $\hat{P}g$ closed map in general as shown in the following example.

Example 3.7: Consider X = Y = Z = {a, b, c} with topologies τ = {∅, a, b, ab, bc, X}, σ = {∅, ab, X} and η = {∅, a, ab, X}. Define f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be the identity mappings. Then f is a $\hat{P}g$ closed map and g is a $\hat{P}g$ closed map but their composition is not a $\hat{P}g$ closed map.

Theorem 3.8: Let (X, τ), (Z, η) be topological spaces and (Y, σ) be a topological space where every $\hat{P}g$ closed subset is closed. Then the composition g o f: (X, τ) → (Z, η) is the $\hat{P}g$ closed map, if f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) are $\hat{P}g$ closed maps.

Proof: Let O be a closed set in X. Since, f is $\hat{P}g$ closed, f (O) is $\hat{P}g$ closed in Y. By hypothesis, f (O) is closed. Since g is $\hat{P}g$ closed, g (f {O}) is $\hat{P}g$ closed in Z and g (f {O}) = g ∘ f (O). Therefore g ∘ f is $\hat{P}g$ closed.

Theorem 3.9: If f: (X, τ) → (Y, σ) is g - closed map and g: (Y, σ) → (Z, η) is $\hat{P}g$ closed map and (Y, σ) is T $_{1/2}$ space. Then the composition g ∘ f: (X, τ) → (Z, η) is $\hat{P}g$ closed map.

Proof: Let O be a closed set in (X, τ). Since f is g – closed, f (O) is g – closed in (Y, σ). Also g is $\hat{P}g$ closed and (Y, σ) is T $_{1/2}$ space which implies g (f (O)) is $\hat{P}g$ closed in Z and g (f (O)) = g ∘ f (O). Therefore, g ∘ f is $\hat{P}g$ closed.

Theorem 3.10: Let f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be two mappings such that their composition g ∘ f: (X, τ) → (Z, η) be $\hat{P}g$ closed mapping. Then the following statements are true.

1. If f is continuous and surjective, then g is $\hat{P}g$ closed.
2. If g is $\hat{P}g$ – irresolute and injective, then f is $\hat{P}g$ closed.
3. If f is g – continuous, surjective and (X, τ) is a $T_{1/2}$ space, then g is $\hat{P}g$ closed.
4. If g is strongly $\hat{P}g$ continuous and injective, then f is $\hat{P}g$ closed.

Proof:

1. Let O be a closed set in (Y, σ). Since, f is continuous, f^{-1}(O) is closed in (X, τ). Since g ∘ f is $\hat{P}g$ closed which implies g ∘ f (f $^{-1}$(O)) is $\hat{P}g$ closed in (Z, η). That is g (O) is $\hat{P}g$ closed in (Z, η), since f is surjective. Therefore, g is $\hat{P}g$ closed.
2. Let O be a closed set in (X, τ). Since g ∘ f is $\hat{P}g$ closed, g ∘ f (O) is $\hat{P}g$ closed in (Z, η). Since g is $\hat{P}g$ – irresolute, g^{-1}(g ∘ f (O)) is $\hat{P}g$ closed in (Y, σ). That is f (O) is $\hat{P}g$ closed in (Y, σ). Since f is injective. Therefore, f is $\hat{P}g$ closed.
3. Let O be a closed set of (Y, σ). Since, f is g – continuous, f $^{-1}$ (O) is g-closed in (X, τ) and (X, τ) is a $T_{1/2}$ space, f $^{-1}$ (O) is closed in (X, τ). Since, g ∘ f is $\hat{P}g$ closed which implies, g ∘ f (f $^{-1}$ (O)) is $\hat{P}g$ closed in (Z, η). That is g (O) is $\hat{P}g$ closed in (Z, η), since f is surjective. Therefore, g is $\hat{P}g$ closed.
4. Let O be a closed set of (X, τ). Since, g ∘ f is $\hat{P}g$ closed which implies, g ∘ f (O) is $\hat{P}g$ closed in (Z, η). Since, g is strongly $\hat{P}g$ continuous, g^{-1} (g ∘ f (O)) is closed in (Y, σ). That is f (O) is closed in (Y, σ). Since g is injective, f is $\hat{P}g$ closed.

4. $\hat{P}g$ Open map

Definition 4.1: A map f: (X, τ) → (Y, σ) is called an **$\hat{P}g$ open** if the image of each open set in X is $\hat{P}g$ open in Y.

Theorem 4.2: A map f: (X, τ) → (Y, σ) is $\hat{P}g$ open if and only if f (int (A)) ⊆ $\hat{P}g$ int (f (A)) for each set A in X.

Proof:

Necessity: Suppose that f is a $\hat{P}g$ open map. Since int (A) ⊆ A, then f (int (A)) ⊆ f (A). By hypothesis, f (int (A)) is $\hat{P}g$ open and $\hat{P}g$ int (f (A)) is the largest $\hat{P}g$ open set contained in f (A). Hence f (int (A)) ⊆ $\hat{P}g$ int (f (A)).

Sufficiency: Suppose A is an open set in X. Then f (int (A)) ⊆ $\hat{P}g$ int (f (A)). Since int (A) = A, then f (A) ⊆ $\hat{P}g$ int (f (A)). Therefore, f (A) is a $\hat{P}g$ open set in (Y, σ) and f is $\hat{P}g$ open map.

Theorem 4.3: Let (X, τ), (Y, σ) and (Z, η) be three topological spaces. f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be two maps. Then

1. If (g o f) is $\hat{P}g$ open and f is continuous, then g is $\hat{P}g$ open map.
2. If (g o f) is open and g is $\hat{P}g$ continuous, then f is $\hat{P}g$ open map.

Proof:

1. Let A be an open set in Y. Then, f $^{-1}$ (A) is an open set in X. Since (g o f) is $\hat{P}g$ open map, then (g o f) (f $^{-1}$ (A)) = g (f ((f $^{-1}$ (A)) = g (A) is $\hat{P}g$ open set in Z. Therefore, g is $\hat{P}g$ open map.
2. Let A be an open set in X. Then, g (f (A)) is an open set in Z. Therefore, g^{-1} (g (f (A))) = f (A) is a $\hat{P}g$ open set in Y. Hence, f is a $\hat{P}g$ open map.

Theorem 4.4: Let f :(X, τ) → (Y, σ) be a bijective map. Then the following are equivalent:

(1) f is a $\hat{P}g$ open map.
(2) f is a $\hat{P}g$ closed map.
(3) f $^{-1}$ is a $\hat{P}g$ continuous map.

Proof:

(1) ⇒ (2) Suppose O is a closed set in X. Then X\O is an open set in X and by (1) f (X\O) is a $\hat{P}g$ open in Y. Since, f is bijective, then f (X\O) = Y\ f (O). Hence f (O) is a $\hat{P}g$ closed in Y. Therefore, f is a $\hat{P}g$ closed map.

(2) ⇒ (3) Let f is a $\hat{P}$g closed map. O be closed set in X. Since, f is bijective then $(f^{-1})^{-1}$ (O) = f (O) which is a $\hat{P}$g closed set in Y. Therefore, f is a $\hat{P}$g continuous map.
(3) ⇒ (1) Let O be an open set in X. Since, f^{-1} is a $\hat{P}$g continuous map then $(f^{-1})^{-1}$ (O) = f (O) is a $\hat{P}$g open set in Y. Hence, f is $\hat{P}$g open map.

Theorem 4.5: A map f :(X, τ) → (Y, σ) is a $\hat{P}$g open if and only if for any subset O of (Y, σ) and any closed set of (X, τ) containing f^{-1}(O), there exists a $\hat{P}$g closed set A of (Y, σ) containing O such that $f^{-1}(A) \subset F$.

Proof: Suppose f is $\hat{P}$g open. Let O ⊂ Y and F be a closed set of (X, τ) such that $f^{-1}(O) \subset F$. Now X - F is an open set in (X, τ). Since f is $\hat{P}$g open map, f (X - F) is $\hat{P}$g open set in (Y, σ). Then A = Y - f (X - F) is a $\hat{P}$g closed set in (Y, σ). Note that $f^{-1}(O) \subset F$ implies O ⊂ A and $f^{-1}(A) = X - f^{-1}(X - F) \subset X - (X - F) = F$. That is, $f^{-1}(A) \subset F$.

Conversely, let B be an open set of (X, τ). Then, $f^{-1}((f(B))^c) \subset B^c$ and $B^c$is a closed set in (X, τ). By hypothesis, there exists a $\hat{P}$g closed set A of (Y, σ) such that $(f(B)^c) \subset A$ and $f^{-1}(A) \subset B^c$ and so $B \subset (f^{-1}(A))^c$. Hence, $A^c \subset f(B) \subset f((f^{-1}(A)))^c$ which implies f (B) = A^c. Since, A^c is a $\hat{P}$g open. f (B) is $\hat{P}$g open in (Y, σ) and therefore f is $\hat{P}$g open map.

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