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Extended Common fixed point theorem for multi-valued mappings in complex valued Metric space

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Abstract: In this paper we are going to prove common fixed point theorem for weak compatible map. We extend the result of (sitthikul and Saejung fixed point Theory and Application 2012:189).the main results announced by Sintunavarat and Kumam (j.inequal. Appl :84,2012).some of the concepts of sequence of function are already given in 2008, Dutta et. al. [7], Rouzkard and Imdad (Comput. Math.appl.,2012,doi:10.1016/j.camwa.2012.02.063). The results announced by sitthikul and Saejung fixed point Theory and Application 2012:189 is mainly improved in this paper.

Keywords: complex valued metric space; multi valued mapping; weak compatible mapping, common fixed point

I. INTRODUCTION

Throughout the article denoted by ℂ is the set of all complex numbers ℕ for set of all natural numbers and ℝ for set of all real numbers.(X,d)(x for short),is a metric space with a metric d.

It is well known that in the literature, there are so many extensions of Banach contraction principle[1],which states that every self-mapping t defined on a complete metric space (x,d) satisfying ,For all,x,y∈X d(Tx,Ty)≤kd(x,y),where k∈[0,1) has unique fixed point for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in}$ℕ is convergent to the fixed point. But the complex valued metric space is a generalization of the classical metric space, introduced by Azam,Fisher and Khan (see [2])

II. PRELIMINARIES

Let us recall a natural relation on ℂ,for $z_1,z_2 \in$ ℂ, define a partial order $\precsim$ on ℂ as follows;

$z_1 \precsim z_2$ iff Re(z1) ≤Re(z2), $Im(z_1) \leq Im(z_2)$

it follows that

$z_1 \precsim z_2$

if one of the following conditions is satisfied:

i. Re(z1)=Re(z2), $Im(z_1)<Im(z_2)$
ii. Re(z1)<Re(z2), $Im(z_1)=Im(z_2)$
iii. Re(z1)<Re(z2), $Im(z_1)<Im(z_2)$
iv. Re(z1)=Re(z2), $Im(z_1)=Im(z_2)$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one the above conditions is not satisfied and we will write $z_1 \prec z_2$ if only iii is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1|<|z_2|,\ z_1 \precsim z_2\ ,\ z_1 \prec z_2 \Rightarrow z_1 \prec z_3$$

Definition 1let X be a nonempty set. A mapping d:XxX→ ℂ is called a complex valued metric on X if the following conditions are satisfied:

(CM1) 0 ≾ d(x,y) for all x,y∈ X and d(x,y)=0⇔x=y.
(CM2) d(x,y)=d(y,x) for all x,y∈ X
(CM3) d(x,y) ≾d(x,z)+d(z,y) for all x,y,z∈ X.

In this case, we say that (X,d) is a complex valued metric space.

It is obvious that this concept is generalization of the classic metric. In fact, if d:XxX→ ℝ satisfies((CM1)-(CM3)), then this d is a metric in the classical sense, that is, the following conditions are satisfies:

(M1) 0 ≤ d(x,y) for all x,y∈ X and d(x,y)=0⇔x=y.
(M2) d(x,y)=d(y,x) for all x,y∈ X
(M3) d(x,y) ≤d(x,z)+d(z,y) for all x,y,z∈ X.

There are so many more different and interesting type of metric spaces and classical theories of metric space for example see[3,4].

Definition 2 Let ℂ be a complex valued metric space,

- We say that a sequence $\{x_n\}$ is said to be a Cauchy sequence be a sequence in x ∈XIf for every c∈ ℂ, with 0≺c there is $n_0 \in$ ℕsuch that for all $n>n_0$such thatd$(x_n,x_m)\prec c$.
- We say that a sequence $\{x_n\}$ converges to an element x∈ X. If for every c∈ ℂ, with 0≺c there exist an integer $n_0 \in$ ℕsuch that for all $n>n_0$such that $d(x_n,x)\prec c$ and we write $x_n \underset{d}{\rightarrow} x$.
- We say that (x,d) is complete if every Cauchy sequence in X converges to a point in X.

The following fact is summarized from Azam,fisher and Khan"s paper[2]. In fact,(b and c of preposition 1.3 are their lemmas 2 and 3.

Preposition 3

Let (X,d) be a complex value metric space. Suppose that $d=d_1+id_2$ where d_1,d_2: XxX→ ℝ, That is,d_1=Re(d) and d_2=Im(d).then the following assertions hold.

a. $|d|=(d_1^2 + d_2^2)^{1/2}$:XxX→ ℝ, is a (classical) metric on X.

b. If $\{x_n\}$ is a sequence in X and x ∈X.Then$x_n \underset{d}{\to} x$. iff $x_n \underset{|d|}{\to} x$.

c. (X,d) is complete if and only if (X,|d|) is complete.

The following common fixed point theorem was also proved by Azam,Fisher and Khan. This can be viewed as a generalization of the well-known Banach fixed Point theorem.

Theorem 4

([2]) Let (X,d) be a complete complex valued metric space, andλ,μ be non-negative real numbers such that λ+μ<1. Suppose that S,T: X→ X are mappings satisfying

$d(Sx,Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Sx)dy,Ty)}{1+d(x,y)}$ for all x,y∈X

then S and T have a unique common fixed point.

As rouzkard and imdad [5] extended and improved the common fixed point theorems which are more general than thaAzam et al.[2]In this paper, we continue the study of comma fixed point theorems and obtain the generalized result proved by sintunavarat and kumam[6] and sitthikul and saejung FTP and applications 2012[7]

III. MAIN RESULT

Lemma 5.let (X,d) be a complex valued metric space and f,S,T: X →X have a unique point of coincidence v in X.if (S,f) and (T,f) are weakly compatible, thens,T,f have a unique common fixed point.

Theorem 6. Let (x,d) be a complex value of matrix space & f, S, T : X→X suppose there exists mappings $g_1, g_2 : x \to [0, 1)$ such that∀ x, y ∈X

i. $g_1(sx) \leq g_1(fx)$ and $g_1(Tx) \leq g_1(fx)$

ii. $g_1(fx) + g_2(fx) + g_3(fx) + g_4(fx) < 1$

iii. $d(Sx,Ty) < g_1(fx)d(fx,fy) + g_2(Sx)d(Sx,Sy) + \frac{g_3(fx)d(fx,Sx)d(fy,Ty)}{(1+d(fx,fy))}$

$$\frac{g_4(sx)d(sx,Tx)d(sy,Ty)}{1 + d(sx,sy)}$$

If S(x)⋃T(x)⊆ f(x) and f(x) is complete, then Sand T have a unique fixed point of coincidence. Moreover, if (S,f) and (T,f) are weakly compatible, then f,S,T have a unique fixed point in X.

Proof:

Let x_0∈ X. Choose x_1∈X such that $Sx_0 = fx_1$ and $Sx_1 = Tx_0$

& $fx_2 = Tx_1$ and $Sx_2 = Tx_1$

Continuing this way we can construct a seq^n $\{fx_n\}$in f(x) such that,

$fx_n= Sx_{n-1}$ if n is odd

$= Tx_{n-1}$ if n is even

&$Sx_n= fx_{n-1}$ if n is odd

Tx_{n-1} if n is even

If n is odd, Then by Hypothesis,

$d(fx_n , fx_{n+1}) = d(Sx_{n-1} , Tx_n)$

$d(Sx_n , Sx_{n+1}) = d(fx_{n-1} , Tx_n)$

since,

$Sx_0 = fx_1$ and $Sx_1 = Tx_0$

& $fx_2 = Tx_1$ and $Sx_2 = Tx_1$

Since,

$d(Sx,Ty) < g_1(fx)d(fx,fy) + g_2(Sx)d(Sx,Sy) + \frac{g_3(fx)d(fx,sx)d(fy,Ty)}{(1+d(fx,fy))} + \frac{g_4(Sx)d(Sx,Tx)d(Sy,Ty)}{1+d(Sx,Sy)}$

$d(Sx_{n-1}, Tx_n) \leq g_1(fx_{n-1})d(fx_{n-1}, fx_n)+ g_2(Sx_{n-1})d(Sx_{n-1}, Sx_n)+ \frac{g_3((fx_{n-1})d(fx_{n-1},Sx_{n-1})d(fx_n,Tx_n)}{1+d(fx_{n-1},fx_n)} + \frac{g_4(Sx_{n-1})d(Sx_{n-1},Tx_{n-1})d(Sx_n,Tx_n)}{1+d(Sx_{n-1},Sx_n)}$

$\leq g_1(fx_{n-1})d(fx_{n-1}, fx_n) + g_2(Sx_{n-1})d(Sx_{n-1}, Sx_n) + \frac{g_3((fx_{n-1})d(fx_{n-1},fx_{n-1})d(fx_n,fx_n)}{1+d(fx_{n-1},fx_n)}$

$+ \frac{g4((Sx_{n-1})d(Sx_{n-1},Sx_{n-1})d(Sx_n,Sx_{n+1})}{1+d(Sx_{n-1},Sx_n)}$

$\therefore d(fx_n, fx_{n+1}) \leq g_1(fx_{n-1})|d(fx_{n-1}, fx_n)|+g_3(fx_{n-1})|d(fx_n, fx_{n+1})| |\frac{d(fx_{n-1},fx_n)}{1+d(fx_{n-1},fx_n)}|$

$+g_2(Sx_{n-1})|d(Sx_{n-1}, Sx_n)|$

$+ g_4(Sx_{n-1})|d(Sx_n, Sx_{n+1})| |\frac{d(Sxn-1,Sxn)}{1+d(Sxn-1,Sxn)}|$

$\leq g_1(fx_{n-1})|dfx_{n-1}, fx_n| + g_3(fx_{n-1})|d(fx_n, fx_{n+1})|$

$+ g_2(Sx_{n-1})|d(Sx_{n-1}, Sx_n) + g_4(Sx_{n-1})|d(Sx_n, Sx_{n+1})|$

$= g_1(Tx_{n-2})|d(fx_{n-1}, fx_n)| + g_3(Tx_{n-2)|}d(fx_n, fx_{n+1})|$

$+ g_2(Tx_{n-2})|d(fx_n, fx_{n+1})|$

$+ g_4(Tx_{n-2})|d(fx_{n-1}, fx_n)|$

$= (g_1 + g_4) (Tx_{n-2})|d(fx_{n-1}, fx_n)| + (g_2 + g_3) (Tx_{n-2})|d(fx_n, fx_{n+1})|$

$\leq (g_1 +g_4) (fx_{n-2})|d(fx_{n-1}, fx_n)| + (g_2 + g_3)(Tx_{n-2})* |d(fx_n, fx_{n+1})$

$= (g_1 + g_4) (Sx_{n-3})|d(fx_{n-1}, fx_n)| +(g_2 + g_3) (Sx_{n-3}) * |d(fx_n , fx_{n+1})|$

$\leq (g_1 + g_4) (fx_{n-3})|d(fx_{n-1}, fx_n)| + (g_2 +g_3)(fx_{n-3})|d(fx_n, fx_{n+1})|$

$= (g_1 + g_4)(Tx_{n-4})|d(fx_{n-1}, fx_n)| + (g_2 + g_3) (Tx_{n-4}) * |d(fx_n, fx_{n+1})|$

$\leq (g_1 + g_4) (fx_{n-4})|d(fx_{n-1}, fx_n)| + (g_2 + g_3) (fx_{n-4}) * |d(fx_n, fx_{n+1})|$

$= (g_1 + g_4) (Sx_{n-5})|d(fx_{n-1}, fx_n)| + (g_2 + g_3) (fx_{n-4}) * |d(fx_n, fx_{n+1})|$

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$\leq (g_1 + g_4) (fx_0)|d(fx_{n-1}, fx_n)| + (g_2 + g_3)(fx_0) * |d(fx_n, fx_{n+1})|$

Which implies that,

$|d(fx_n, fx_{n+1})| \leq \frac{(g_1 +g_4)(fx)d(fx_{n-1},fx_n)}{1+(g_2 +g_3)(fx)}$

If n is even,

$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Sx_n) = d(Sx_n, Tx_{n-1})$

$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, fx_n) = d(fx_n, Tx_{n-1})$

$d(fx_n, fx_{n+1}) = d(x_{n-1}, Sx_n) = d(Sx_n, Tx_{n-1})$

$\leq g_1(fx_n)d(fx_n, fx_{n-1}) + g_2(Sx_n)d(Sx_n, Sx_{n-1})$

$+ g_3(fx_n)d(fx_n, Sx_n)d(fx_{n-1}, Tx_{n-1}) / 1+d(fx_n, fx_{n-1})$

$+ g_4(Sx_n)d(Sx_n, Tx_n)d(Sx_{n-1}, Tx_{n-1}) / + d(Sx_n, Sx_{n-1})$

Therefore,

$| d(fx_n, fx_{n+1})| \leq g_1(fx_n)|d(fx_n, fx_{n-1})| + g_2(Sx_n)|d(Sx_n, Sx_{n-1})|$

$+ g_3(fx_n)d(fx_n, fx_{n+1})|d(fx_{n-1}, Tx_{n-1}) / 1+d(fx_n, fx_{n-1})|$

$+ g_4(Sx_n)|d(Sx_n, Sx_{n+1})|d(Sx_n, Sx_{n-1}) / 1+d(Sx_{n-1}, Sx_n)|$

$\leq g_1(fx_n)|d(fx_{n-1},fx_n)| + g_3(fx_n)|d(fx_n, fx_{n+1})|$

$+g_2(Sx_n)|d(Sx_n, Sxn_{-1})| + g_4(Sx_n)|d(Sx_n, Sx_{n+1})|$

$= g_1(Tx_n)|d(fx_{n-1},fx_n)| + g_3(Tx_n)|d(fx_n, fx_{n+1})|$

$+g_2(Tx_n)|d(fx_n, fxn_{-1})| + g_4(Tx_n)|d(fx_n, fx_{n+1})|$

$=(g_{1+} g_{4)}(Tx_n)|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(Tx_n)|d(fx_n, fx_{n+1})|$

$\leq(g_{1+} g_{4)}(fx_{n-1})|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(fx_{n-1})|d(fx_n, fx_{n+1})|$

$=(g_{1+} g_{4)}(Sx_n)|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(Sx_{n-2})|d(fx_n, fx_{n+1})|$

$\leq(g_{1+} g_{4)}(fx_{n-2})|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(fx_{n-2})|d(fx_n, fx_{n+1})|$

$=(g_{1+} g_{4)}(Tx_{n-3})|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(Tx_{n-3})|d(fx_n, fx_{n+1})|$

$\leq(g_{1+} g_{4)}(fx_{n-3})|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(fx_{n-3})|d(fx_n, fx_{n+1})|$

$\leq(g_{1+} g_{4)}(Sx_{n-4})|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(fx_{n-4})|d(fx_n, fx_{n+1})|$

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$\leq(g_{1+} g_{4)}(fx_{n0})|d(fx_{n-1},fx_n)| + (g_2+g_{3)}(fx_{n0})|d(fx_n, fx_{n+1})|$

Which implies that

(1) $|d(fx_n, fx_{n+1})| \leq \frac{(g_1 +g_4)(fx)d(fx_{n-1},fx_n)}{1+(g_2 \ +g_3)(fx)}$

Let $\alpha=\frac{(g_1 +g_4)(fx)d(fx_{n-1},fx_n)}{1+(g_2 \ +g_3)(fx)}$

By repeating application of (1)

$|d(fx_n, fx_{n+1})| \leq \alpha| d(fx_{n-1,}fx_n)|$

$\leq \alpha^2|d(fx_{n-2,} fx_{n-1})|$

$\leq\alpha^3|d(fx_{n-3,}fx_{n-2})|$

$\leq \alpha^4|d(fx_{n-4,} fx_{n-3})|$

$\leq \alpha^5|d(fx_{n--5,} fx_{n-4})|$

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$\leq \alpha^n| d(fx_{0,} fx_1)|$

For all n,m∈IN, m>n,

$d(fx_n,fx_m) \leq |d(fx_n, fx_{n+1})|+ |d(fx_{n+1}, fx_{n+2})| + |d(fx_{n+2}, fx_{n+3})|+\ldots\ldots\ldots\ldots + |d(fx_{m-1}, fx_m)|$

hence,

$\leq(\alpha^n +\alpha^{n+1} + \alpha^{n+2} +\ldots\ldots\ldots\ldots\ldots+\alpha^{m-1}) | d(fx_{0,} fx_1)|$

Since

$\alpha\in[0,1)$, $\lim_{n\to\infty}$ we have $| d(fx0, fx1)|$ approaches to zero.

which imply that $fx_n$is a Cauchy sequence, by completeness of fx, there exist u,v∈X such that $fx_n\to v = fu$

$d(fu,Tu) \leq d(fu,fx_{2n+1})+d(fx_{2n+1},Tu)$

$= d(fu,fx_{2n+1})+d(Sx_{2n},Tu)$

$\leq d(fu,fx_{2n+1})+g_1 (fx_{2n+1})d(fx_{2n},fu)+ g_2 (Sx_{2n})d(Sx_{2n},Su)+ \frac{g_3(fx_{2n})d(fx_{2n},Sx_{2n})d(fu,Tu)}{1+d(fx_{2n},fu)} + \frac{g_4(Sx_{2n})d(Sx_{2n},Tx_{2n})d(Su,Tu)}{1+d(Sx_{2n},Su)}$

Which implies that

$|d(fu,Tu)|\leq| d(fu,fx_{2n+1})|+g_1 (fx_{2n})|d(fx_{2n},fu)|+ g_2 (Sx_{2n})|d(Sx_{2n},Su)|+ \frac{g_3(fx_{2n})|d(fx_{2n},Sx_{2n})||d(fu,Tu)|}{|1+d(fx_{2n},fu)|} + \frac{g_4(Sx_{2n})|d(Sx_{2n},Tx_{2n})||d(su,Tu)|}{|1+d(fx_{2n},fu)|}$

$\leq |d(fu,fx_{2n+1})|+g_1 (fx_{2n})|d(fx_{2n},fu|)+ g_2 (Sx_{2n})|d(Sx_{2n},Su)|+ \frac{g_3(fx_{2n})|d(fx_{2n},Sx_{2n})||d(fu,Tu)|}{1} + \frac{g_4(Sx_{2n})|d(Sx_{2n},Tx_{2n})||d(Su,Tu)|}{1}$

Since

$1\leq 1+d(fx_{2n},fu)$

$1\leq 1+d(Sx_{2n},Su)$

$\leq|d(fu,fx_{2n+1})|+g_1 (fx_0)|d(fx_{2n},fu)|+ g_2 (Sx_0)|d(Sx_{2n},Su)|+ g_3(fx_0)|d(fx_{2n},Sx_{2n+1})||d(fu,Tu)|+ g_4(sx_0)|d(Sx_{2n}, Sx_{2n+1})||d(Su,Tu)|$

If $n\to\infty$,$|d(fu,Tu)|\to0$,hence $d(fu,Tu)\to0$

Implies,fu=tu=v,similarlyfu=su=v

Su=Tu

Thus,fu=Su=Tu=v and v become a common fixed point of coincidence of f,S and T.

Uniqueness, Let there exist w(≠v)∈X such that fx=Sx=Tx=w sor some x∈X. Thus,

$d(v,w)=d(Su,Tx)$

$\leq g_1(fu)d(fu,fx)+g_2(Su)d(Su,Sx)+ \frac{g_3(fu)d(fu,Su)d(fx,Tx)}{1+d(fu,fx)}+\frac{g_4(fu)d(fu,Tu)d(fx,Sx)}{1+d(Su,Sx)}$

$\leq g_1(v)d(v,w)+g_2(v)d(v,w)\frac{g_3(v)d(v,v)d(w,w)}{1+d(v,w)}+ \frac{g_4(v)d(v,v)d(w,w)}{1+d(w,w)}$

$=g_1(v)d(v,w)$

Implies

$|d(v,w)|\leq g_1(v)d(v,w)$

Since, $g_1\epsilon[0,1)$

$|d(v,w)\to0$

So, v=w.if (S,f) and (T,f) are weakly compatible, then by lemma (3.1),f,S,T have a unique common fixed point in X

Corollary 7 .

Let (x,d) be a complex valued of matrix space & f, T : X →X satisfying∀ x, y ∈X

$d(Tx,Ty)<\lambda d(fx,fy) \quad +\frac{\mu d(fx,Tx)d(fy,Ty)}{1+d(fx,fy)} + \frac{\gamma d(Tx,Ty)d(fy,fy)}{1+d(Tx,Ty)}$

for all x,y∈ X, where μ,λ are non-negative real numbers with μ+λ<1. If T(x)⊆ f(x) and f(x) is complete, then f andT have a unique pint of coincidence. Moreover, if f and T are weakly compatible, then f and T have common fixed point in X.

proof. We can prove this result by settingS=T,$g_1(x)=\lambda$,$g_2(x)=\mu$ in theorem 6.

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