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Product Cordial Labeling Of Product Graphs

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***Abstract*: Let G = (V,E) be a graph. A binary vertex labeling f: V(G) → {0,1} of a graph G with induced edge labeling f*: E(G) → {0,1} defined by f*(e=uv) = f(u)f(v) is said to be product cordial if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ represents the number of vertices and edges labeled i for i = 0,1. A graph G is product cordial if it admits a product cordial labeling. In this paper, we analyse some special and product graphs for the existence of product cordial labeling.**

Keywords: Product cordial labeling, Cartesian product, Weak product.

AMS Subject classification: 05C78

I INTRODUCTION

Let G = (V(G),E(G)) be a finite, undirected simple graph. A graph labeling [3] is an assignment of values to the vertices of the graph satisfying certain conditions. A ***binary vertex labeling*** of G is simply a function f: V(G) → {0,1}. Here, f(v) is said to be the label of v. Let the ***induced edge labeling*** f*: E(G) → {0,1} be defined by f*(e=uv) = $|f(u) - f(v)|$. Let $v_f(i)$ and $e_f(i)$be the number of vertices and edges labeled i for i = 0,1. A binary vertex labeling is called a ***cordial labeling*** of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.A graph G is cordial if it admits a cordial labeling. The concept of cordial labeling was introducted by I.Cahit[2]. A binary vertex labeling of a graph G with induced edge labeling f*: E(G) →{0,1} defined by f*(e=uv) = f(u)f(v) is called a ***product cordial labeling*** if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ are as earlier. A graph G is product cordial if it admits a product cordial labeling[6].***Ladder graph*** L_n[7] is a planar undirected graph with 2n vertices and 3n−2 edges which is actually the Cartesian product of P_2 and P_n. ***Ladder rung graph*** LR_n[7] is the graph union of n copies of P_2.Let G_1 = (V_1,E_1) and G_2 = (V_2,E_2) be two graphs. The ***middle graph*** M(G) [5]of a graph G is the graph whose vertex set is V(G)∪E(G) and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is vertex of G and the other is an edge incident with it.The ***cartesian product***[1] of G_1 and G_2 denoted by $G_1 \times G_2$ has V = $V_1 \times V_2$ as its vertex set E=$\left\{\begin{matrix}(u_1, v_1), (u_2, v_2)/u_1 = u_2 \text{and } v_1v_2 \in E_2 \\ or \quad v_1 = v_2 \text{and} u_1u_2 \in E_1\end{matrix}\right\}$. The ***weak (or kronecker) product***[4] of G_1 and G_2 denoted by $G_1 \odot G_2$ has V=$V_1 \times V_2$ as its vertex set and E={{(u_1,v_1),(u_2,v_2)}/(u_1,u_2)∈E_1 and (v_1,v_2)∈E_2} as its edge set. In this paper, we analyse some special and product graphs for the existence of product cordial labeling.

1.1 Theorem:[6] P_n is product cordial.

II PRODUCT CORDIAL LABELING OF SOME SPECIAL GRAPHS

2.1 Theorem: LR_n is product cordial.

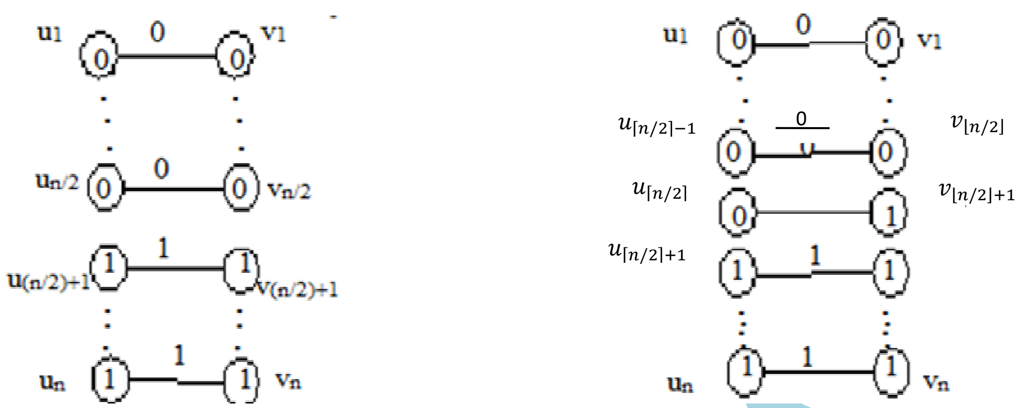
Proof: $LR_n \cong nP_2$

Let V(LR_n) = {u_i,v_i/ i = 1,..,n}where u_i,v_i are the end vertices of i^{th} copy of P_2.

Now, label n vertices with 0 and n vertices with 1 as in figure 2.1

Therefore, $v_f(0) = v_f(1)$ = n.

Thus, $|v_f(0) - v_f(1)| = 0 < 1$ → (1)



n is even n is odd

Figure 2.1

Case i) n is odd

Here, f : V(LR_n) → {0,1}isdefined by

$$f(u_i) = \begin{cases} 0 & if\ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 1 & if\ \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases} \text{ and}$$

$$f(v_i) = \begin{cases} 0 & if\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 1 & if\ \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n \end{cases}$$

Correspondingly, the edges of $\left\lfloor \frac{n}{2} \right\rfloor$ copies of P_2 get the label 1 and the edges of $\left\lceil \frac{n}{2} \right\rceil$copies of P_2 get the label 0.

Therefore, $e_f(0) = \left\lceil \frac{n}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor$.

Thus,$|e_f(0) - e_f(1)| = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor = 1 \rightarrow (2)$

By (1) and (2), LR_n is product cordial.

Case ii) n is even

Here, f : V(LR_n) → {0,1} is defined by

$$f(u_i) = f(v_i) = \begin{cases} 0 & if\ 1 \le i \le (n/2) \\ 1 & if\ \left(\frac{n}{2}\right) + 1 \le i \le n \end{cases}$$

Correspondingly, the edges of (n/2) copies of P_2 get the label 0 and the edges of (n/2) copies of P_2 get the label 1.

Therefore, $e_f(0) = e_f(1)$ = (n/2).

Thus,$|e_f(0) - e_f(1)| = e_f(0) \sim e_f(1) = 0 < 1 \rightarrow (3)$

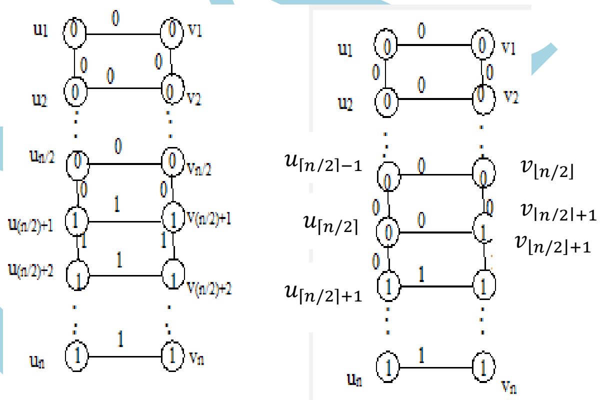
By (1) and (3), LR_n is product cordial.

2.2 Theorem: L_n is product cordial iff n = 1.

Proof: L_n has 2n vertices. Let V(L_n) = {$u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n$} with u_i,v_i representing the corresponding column elements.

Now, label n vertices with 0 and n vertices with 1 as in figure 2.2

Therefore, $v_f(0) = v_f(1)$ = n. Thus, $|v_f(0) - v_f(1)| = 0 < 1 \rightarrow (1)$



n is even n is odd

Figure 2.2

Case i) n is odd

When n = 1, $L_n \cong P_2$.

By 1.1, L_n(n=1) is product cordial.

Let n = 2k + 1, k = 1,2,3,…

Let f be a vertex labeling satisfying (1).

Equation (2) defines one such f which gives maximum value for $e_f(1)$

$$f(u_i) = \begin{cases} 0 & if\ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 1 & if\ \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases} \text{ and}$$

$$f(v_i) = \begin{cases} 0 & if\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 1 & if\ \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n \end{cases} \rightarrow (2)$$

Therefore, $e_f(1) \le$ n + k−2 and so$e_f(0) \ge$ (3n−2) – (n+ k−2) = 2n – k.

Thus,$|e_f(0) - e_f(1)| = e_f(0) \sim e_f(1)$

$\ge |2n - k - (n + k - 2)|$

$= |n - 2k + 2|$

$= |2k + 1 - 2k + 2| = 3.$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Since f assigns maximum value for $e_f(1)$, there is no other function which is a product cordial labeling of L_n when n is odd.

Case ii) n is even

Let n = 2k, k = 1,2,3,…

Let f be a vertex labeling satisfying (1).

Equation (3) defines one such f which gives maximum value for $e_f(1)$

f(u_i) = f(v_i) = $\begin{cases} 0 & if\ 1 \le i \le (n/2) \\ 1 & if\ \left(\frac{n}{2}\right) + 1 \le i \le n \end{cases}$ → (3)

Therefore, $e_f(1) \le$ n+k−2 and so $e_f(0) \ge$ (3n−2) – (n + k−2) = 2n – k.

Thus, $|e_f(0) - e_f(1)| = e_f(0) \sim e_f(1)$
$\ge$ |2n – k – (n + k – 2)|
= |n – 2k + 2|
= |2k – 2k + 2| = 2.

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Hence, as in case (i), there is no other function which is a product cordial labeling of L_n when n is even.

By cases (i) & (ii), L_n is product cordial iff n = 1.

2.3 Theorem: M(C_n)is not product cordial.

Proof:M(C_n) has 2n vertices.

Let V(M(C_n))={v_i,u_i/ i=1,…,n} where V(C_n) = { v_i / i=1,…,n} and E(C_n) = { u_i / i=1,…,n}

Correspondingly,E(M(C_n))={u_iu_{i+1}/i=1,…,n-1} ∪{u_nu_1}∪{v_iu_i,v_iu_{i+1}/i=1,…,n-1}∪{v_nu_n,v_nu_1}

The following figure 2.3 represents M(C_n) for general n.

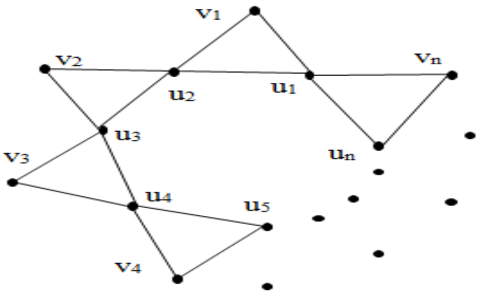


Figure 2.3

Now, label n vertices with 0 and n vertices with 1, so that $|v_f(0) - v_f(1)| \le 1$ → (1)

Case i) n is odd

Let n = 2k+1, k= 1,2,3,…

Equation (2) defines a vertex labeling f satisfying (1) & gives maximum value for$e_f(1)$.

f(u_i) = $\begin{cases} 1 & if\ 1 \le i \le \lceil n/2 \rceil \\ 0 & if\ \lceil n/2 \rceil + 1 \le i \le n \end{cases}$ and

f(v_i) = $\begin{cases} 1 & if\ 1 \le i \le \lfloor n/2 \rfloor \\ 0 & if\ \lfloor n/2 \rfloor + 1 \le i \le n \end{cases}$ → (2)

Therefore, $e_f(1) \le n + k - 1$ and so

$e_f(0) \ge (3n) - (n + k - 1) = 2n - k + 1$.

Thus,

$|e_f(0) - e_f(1)| \ge 2n - k + 1 - (n + k - 1)$
$= n - 2k + 2 = 2k + 1 - 2k + 2$
$= 3$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Since f assigns maximum value for $e_f(1)$, there is no other function which is a product cordial labeling of M(C_n)when n is odd.

Case ii) n is even

Let n = 2k, k= 2,3,…

Equation (3) defines a vertex labeling f satisfying (1) &gives maximum value for $e_f(1)$.

f(u_i) = f(v_i) = $\begin{cases} 0 & if\ 1 \le i \le \left(\frac{n}{2}\right) \\ 1 & if\ \left(\frac{n}{2}\right) + 1 \le i \le n \end{cases}$ → (3)

Therefore, $e_f(1) \le n + k - 2$ and so

$e_f(0) \ge (3n) - (n + k - 2) = 2n - k + 2$.

Thus,

$|e_f(0) - e_f(1)| \ge 2n - k + 2 - (n + k - 2)$
$= n + 4 - 2k$
$= 2k + 4 - 2k = 4$.

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Hence, as in case (i), there is no other function which is a product cordial labeling of M(C_n)when n is even.

By cases (i) & (ii),M(C_n) is not product cordial.

II PRODUCT CORDIAL LABELING OF CARTESIAN PRODUCT GRAPHS

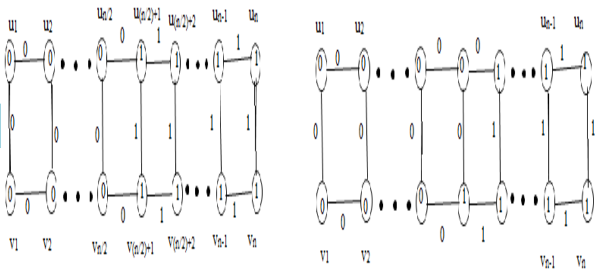
We restate the definition of **Cartesian Product**[4] as follows: Let G_1 = (U,E) and G_2 = (V,E') be two graphs. Let U = {u_1,u_2,…,u_n} and V = {v_1,v_2,…,v_m}. The Cartesian product of G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set W = U×V and edge set E'' = {{(u_i,v_j),(u_k,v_s)}/u_i = u_k and v_j is adjacent to v_s or v_j = v_s and u_i is adjacent to u_k}.

3.1 Theorem: $P_2 \times P_n$ is product cordial iff n=1.

Proof:$P_2 \times P_n$ has 2n vertices.

$P_2 \times P_n$ looks as in figure 3.1

Let V($P_2 \times P_n$) = {u_1,u_2,…,u_n,v_1,v_2,…,v_n} with u_i,v_i representing 1st and 2nd row elements respectively.



n is even

n is odd

Figure 3.1

Now, label n vertices with 0 and n vertices with 1, so that $|v_f(0) - v_f(1)| = 0 < 1$ → (1)

Case i) n is odd

When n=1, $P_2 \times P_n \cong P_2$

By 1.1, $P_2 \times P_n$ is product cordial

Let n = 2k+1, k= 1,2,3,…

Equation (2) defines a vertex labeling f satisfying (1) & gives maximum value for $e_f(1)$.

f(u_i) = $\begin{cases} 0 & if\ 1 \le i \le \lceil n/2 \rceil \\ 1 & if\ \lceil n/2 \rceil + 1 \le i \le n \end{cases}$ and

f(v_i) = $\begin{cases} 0 & if\ 1 \le i \le \lfloor n/2 \rfloor \\ 1 & if\ \lfloor n/2 \rfloor + 1 \le i \le n \end{cases}$ → (2)

Therefore, $e_f(1) \le n + k - 2$ and so

$e_f(0) \ge (3n - 2) - (n + k - 2)$
$= 3n - n - 2 - k + 2 = 2n - k$.

Thus,

$$|e_f(0) - e_f(1)| \geq 2n - k - (n + k - 2)$$
$$= 2k + 1 - 2k + 2 = 3$$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Since f assigns maximum value for $e_f(1)$, there is no other function which is a product cordial labeling of $P_2 \times P_n$ when n is odd.

Case ii) n is even

Let n = 2k, k= 1,2,3,…

Equation (3) defines a vertex labeling f satisfying (1) & gives maximum value for $e_f(1)$.

$$f(u_i) = f(v_i) = \begin{cases} 0 & if\ 1 \leq i \leq (\frac{n}{2}) \\ 1 & if\ (\frac{n}{2}) + 1 \leq i \leq n \end{cases} \rightarrow (3)$$

Therefore, $e_f(1) \leq n + k - 2$ and so

$e_f(0) \geq (3n - 2) - (n + k - 2)$

$= 3n - n - 2 - k + 2 = 2n - k.$

Thus, $|e_f(0) - e_f(1)| \geq 2n - k - (n + k - 2) = 2k - 2k + 2 = 2.$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Hence, as in case (i), there is no other function which is a product cordial labeling of $P_2 \times P_n$ when n is even.

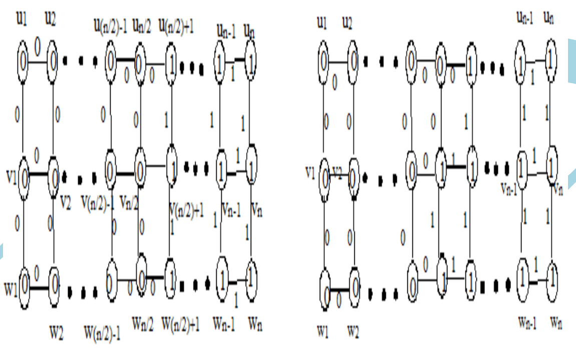
By cases (i) & (ii), $P_2 \times P_n$ is product cordial iff n =1.

3.2 Theorem: $P_3 \times P_n$ is product cordial iff n=1.

Proof: $P_3 \times P_n$ has 3n vertices.

$P_3 \times P_n$ looks as in figure 3.2

Let $V(P_3 \times P_n)=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_n\}$ with u_i,v_i,w_i representing the 1st, 2nd 3rd row elements respectively.



n is even **n is odd**

Figure 3.2

Case i) n is odd

When n=1, $P_3 \times P_n \cong P_3$

By 1.1, $P_3 \times P_n$ is product cordial

Let n = 2k+1, k= 1,2,3,…

Now, label $\lceil 3n/2 \rceil$ vertices with 0 and $\lfloor 3n/2 \rfloor$ vertices with 1, so that $|v_f(0) - v_f(1)| = 1 \rightarrow (1)$

Equation (2) defines a vertex labeling f satisfying (1)& gives maximum value for $e_f(1)$.

$$f(u_i) = \begin{cases} 0 & if\ 1 \leq i \leq \lceil n/2 \rceil \\ 1 & if\ \lceil n/2 \rceil + 1 \leq i \leq n \end{cases} \text{ and}$$

$$f(v_i) = f(w_i) = \begin{cases} 0 & if\ 1 \leq i \leq \lfloor n/2 \rfloor \\ 1 & if\ \lfloor n/2 \rfloor + 1 \leq i \leq n \end{cases} \rightarrow (2)$$

Therefore, $e_f(1) \leq 2n + k - 2$ and $e_f(0) \geq (5n - 3) - (2n + k - 2) = 3n - k - 1.$

Thus, $|e_f(0) - e_f(1)| \geq 3n - k - 1 - (2n + k + 2)$

$= n - 2k + 1 = 2k + 1 - 2k + 1 = 2$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Since f assigns maximum value for $e_f(1)$, there is no other function which is a product cordial labeling of $P_3 \times P_n$ when n is odd.

Case ii) n is even

Let n = 2k, k= 1,2,3,…

Now, label 3n/2 vertices with 0 and 3n/2 vertices with 1, so that $|v_f(0) - v_f(1)| = 0 < 1 \rightarrow (3)$

Equation (4) defines a vertex labeling f satisfying (1) & gives maximum value for $e_f(1)$.

$$f(u_i) = f(v_i) = f(w_i) = \begin{cases} 0 & if\ 1 \leq i \leq (\frac{n}{2}) \\ 1 & if\ (\frac{n}{2}) + 1 \leq i \leq n \end{cases} \rightarrow (4)$$

Therefore, $e_f(1) \leq n + 3k - 3$ and so

$e_f(0) \geq (5n - 3) - (n + 3k - 3)$

$= 5n - n - 3 - 3k + 3 = 4n - 3k$

Thus, $|e_f(0) - e_f(1)| \geq 4n - 3k - (n + 3k - 3)$

$= 3n - 3 - 6k = 3(2k) - 3 - 6k = 3.$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Hence, as in case (i), there is no other function which is a product cordial labeling of $P_3 \times P_n$ when n is even.

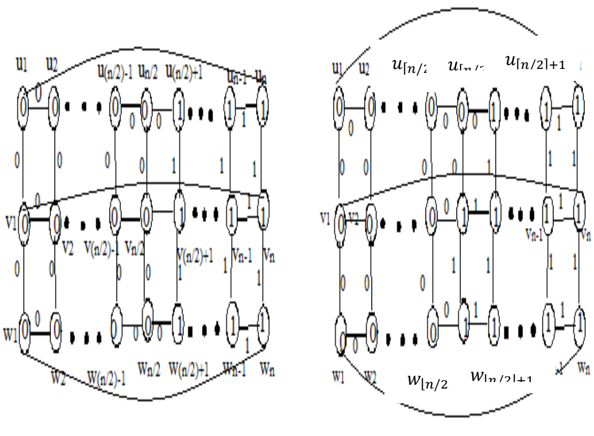
By cases (i) & (ii), $P_3 \times P_n$ is product cordial iff n =1.

3.3Theorem: $C_3 \times C_n$ is not product cordial.

Proof: $C_3 \times C_n$ has 3n vertices.

$C_3 \times C_n$ looks as in figure 3.3

Let $V(C_3 \times C_n)=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_n\}$ with u_i,v_i,w_i representing the 1st,2nd and 3rd row elements respectively.



n is even **n is odd**

Figure 3.3

Case i) n is odd

Let n = 2k+1, k= 1,2,3,…

Now, label $\lceil 3n/2 \rceil$ vertices with 0 and $\lfloor 3n/2 \rfloor$ vertices with 1, so that $|v_f(0) - v_f(1)| = 1.$

Equation (2) defines a vertex labeling f satisfying (1) &gives maximum value for $e_f(1)$.

$f(u_i) = \begin{cases} 0 & if\ 1 \le i \le \lceil n/2 \rceil \\ 1 & if\ \lceil n/2 \rceil + 1 \le i \le n \end{cases}$ and

$f(v_i) = f(w_i) = \begin{cases} 0 & if\ 1 \le i \le \lfloor n/2 \rfloor \\ 1 & if\ \lfloor n/2 \rfloor + 1 \le i \le n \end{cases}$

Therefore, $e_f(1) \le 2n + k - 2$ and $e_f(0) \ge (5n) - (2n + k - 2) = 3n - k + 2$.

Thus, $|e_f(0) - e_f(1)| \ge 3n - k + 2 - (2n + k - 2)$

$= n - 2k + 4$

$= 2k + 1 - 2k + 4 = 5$

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Since f assigns maximum value for $e_f(1)$, there is no other function which is a product cordial labeling of $C_3 \times C_n$ when n is odd.

Case ii) n is even

Let n = 2k, k= 2,3,…

Now, label 3n/2 vertices with 0 and 3n/2 vertices with 1, so that $|v_f(0) - v_f(1)| = 0 < 1$.

Equation (4) defines a vertex labeling f satisfying (3) & gives maximum value for $e_f(1)$.

$f(u_i) = f(v_i) = f(w_i) = \begin{cases} 0 & if\ 1 \le i \le (\frac{n}{2}) \\ 1 & if\ (\frac{n}{2}) + 1 \le i \le n \end{cases} \rightarrow (4)$

Therefore, $e_f(1) \le n + 3k - 3$ and so

$e_f(0) \ge (5n) - (n + 3k - 3) = 4n - 3k + 3$.

Thus,

$|e_f(0) - e_f(1)| \ge 4n - 3k + 3(n + 3k - 3)$

$= 3n + 6 - 6k = 3(2k) + 6 - 6k$

$= 6$.

Therefore, $|e_f(0) - e_f(1)| \nleq 1$

Hence, as in case (i), there is no other function which is a product cordial labeling of $C_3 \times C_n$ when n is even.

By cases (i) &(ii), $C_3 \times C_n$ is not product cordial.

III PRODUCT CORDIAL LABELING OF WEAK PRODUCT GRAPHS

We restate the definition of **Weak(or Kronecker) Product**[4]as follows: Let G_1 = (U,E) and G_2 = (V,E') be two graphs. Let U = {$u_1,u_2,\ldots,u_n$} and V = {$v_1,v_2,\ldots,v_m$}. The weak product of G_1 and G_2 is the graph $G_1 \odot G_2$ with vertex set W = U×V and edge set E" = {{$(u_i,v_j),(u_k,v_s)$}/u_i is adjacent to u_k and v_j is adjacent to v_s}.

4.1 Theoem: $P_2 \odot P_n$ is product cordial.

Proof: $P_2 \odot P_n$ has 2n vertices.

Let $V(P_2)$ = {u,v} and $V(P_n)$ = {$w_1,w_2,\ldots w_n$}. Then, $V(P_2 \odot P_n)$ = {$(u,w_i),(v,w_i)$/ i =1,2,..,n}. Now, name the vertices (u,w_i) as u_i and (v,w_i) as v_i as in figure 4.1

Therefore, $V(P_2 \odot P_n)$ = {u_i,v_i/ i =1,2,..,n}

Here, f: $V(P_2 \odot P_n) \rightarrow$ {0,1} is defined by

$f(u_i) = \begin{cases} 0 & if\ i\ is\ odd \\ 1 & if\ i\ is\ even \end{cases}$, $f(v_i) = \begin{cases} 0 & if\ i\ is\ even \\ 1 & if\ i\ is\ odd \end{cases}$

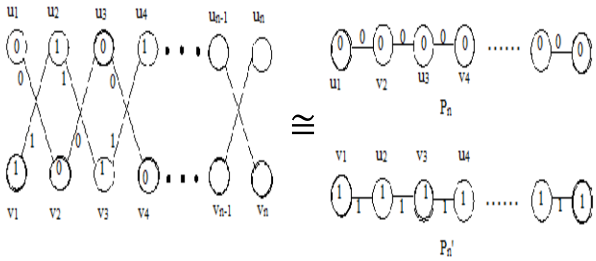


Figure 4.1

From figure 4.1, $P_2 \odot P_n$ is the union of two disjoint paths of length n.ie, $P_2 \odot P_n = P_n \cup P_n'$.

Now, label n vertices of P_n with 0 and label n vertices of P_n' with 1, so that

$|v_f(0) - v_f(1)| = 0 < 1 \rightarrow (2)$

Correspondingly, $e_f(0) = e_f(1)$ = n−1.

Therefore, $|e_f(0) - e_f(1)| = 0 < 1 \rightarrow$ (2)

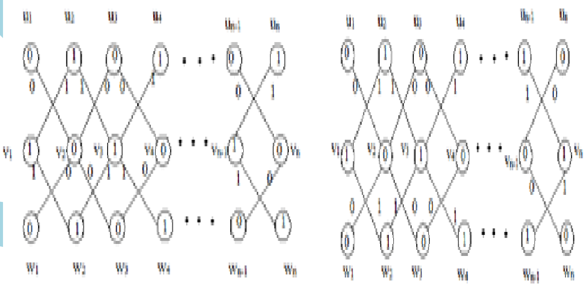
By (1) and (2), $P_2 \odot P_n$ is product cordial.

4.2 Theorem: $P_3 \odot P_n$ is product cordial.

Proof: $P_3 \odot P_n$ has 3n vertices.

$P_3 \odot P_n$ looks as in figure 4.2

Let $V(P_3 \odot P_n)$ = {$u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_n$} with u_i,v_i,w_i representing the 1st ,2nd and 3rd row elements respectively.



n is even **n is odd**

Figure 4.2

Here, f: $V(P_3 \odot P_n) \rightarrow$ {0,1}is defined by

$f(u_i) = \begin{cases} 0 & if\ i\ is\ odd \\ 1 & if\ i\ is\ even \end{cases}$, $f(v_i) = \begin{cases} 0 & if\ i\ is\ even \\ 1 & if\ i\ is\ odd \end{cases}$

and $f(w_i) = \begin{cases} 0 & if\ i\ is\ odd \\ 1 & if\ i\ is\ even \end{cases}$

Case i) n is odd

Now, $v_f(0) = \lceil \frac{3n}{2} \rceil$ and $v_f(1) = \lfloor \frac{3n}{2} \rfloor$ so that $|v_f(0) - v_f(1)| = 1 \rightarrow (1)$

Correspondingly, all the edges incident with u_i, w_i for i = 2,4,…,n-1 get the label 1.

From figure 4.2, it is clear that two edges are incident with u_i,w_i for i = 2,4,…,n−1.

Therefore, $e_f(1) = 2\{(2+2+\ldots\frac{n-1}{2}\text{times})\}$

$= 2\{2(\frac{n-1}{2})\} = 2(n-1) = 2n-2$.

Further, $e_f(0) = q - e_f(1)$ = (4n−4) – (2n−2)

= 2n− 2.

Thus, $|e_f(0) - e_f(1)| = 0 < 1 \rightarrow$ (2)

By (1) and (2), $P_3 \odot P_n$ is product cordial.

Case ii) n is even

Now, $v_f(0)$ = 3n/2 and $v_f(1)$ = 3n/2 so that $|v_f(0) - v_f(1)| = 0 < 1$ → (3)

Correspondingly, all the edges incident with u_i, w_i for i = 2,4,…,n get the label 1.

From figure 4.2, it is clear that two edges are incident with u_i,w_i for i = 2,4,…,n−2 and exactly one edge with u_n and w_n and these edges are all independent.

Therefore, $e_f(1) = 2\{(2+2+\ldots\frac{n-2}{2}\text{times}) + 1\}$

$= 2\{2(\frac{n-2}{2}) + 1\} = 2(n-2) + 2$

$= 2n - 2.$

Further, $e_f(0) = q - e_f(1) = (4n-4) - (2n-2)$

$= 2n - 2.$

Thus, $|e_f(0) - e_f(1)| = 0 < 1$ → (4)

By (3) and (4), $P_3 \odot P_n$ is product cordial.

4.3 Theorem: $P_4 \odot P_n$ is product cordial.

Proof: $P_4 \odot P_n$ has 4n vertices.

$P_4 \odot P_n$ looks as in figure 4.3

Let $V(P_4 \odot P_n) = \{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_n, z_1,z_2,\ldots,z_n\}$ with u_i,v_i,w_i,z_i representing the 1st, 2nd, 3rd and 4th row elements respectively.

Here, f: $V(P_4 \odot P_n) \to \{0.1\}$ is defined by

$f(u_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$,

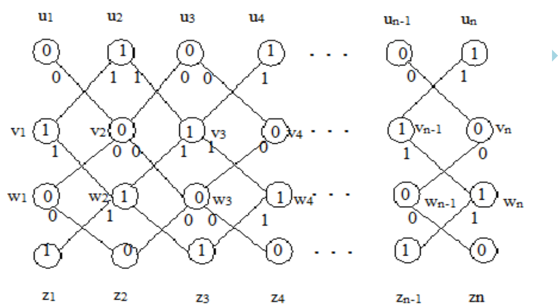
$f(v_i) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$,

$f(w_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$ and

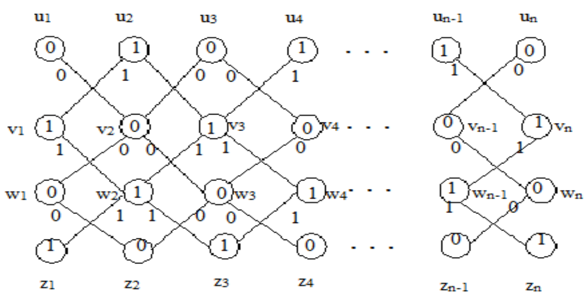
$f(z_i) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$

For any n, we can label 2n vertices with 0 and 2n vertices with 1, so that

$|v_f(0) - v_f(1)| = 0 < 1$ → (1)



n is even



n is odd

Figure 4.3

Case i) n is odd

Correspondingly, the edges incident with u_i, w_i for i = 2,4,…,n-1 get the label 1.

From figure 4.3, it is clear that two edges are incident with u_i for i = 2,4,…,n-1 and four edges incident with w_i for i = 2,4,…,n-1. Further, these edges are all distinct.

Therefore,

$e_f(1) = \{(2+2+\ldots\frac{n-1}{2}\text{times}) + (4+4+\ldots\frac{n-1}{2}\text{times})\}$

$= 2(\frac{n-1}{2}) + 4(\frac{n-1}{2}) = (n-1) + 2(n-1)$

$= 3n - 3.$

Further, $e_f(0) = q - e_f(1)$

$= (6n-6) - (3n-3) = 3n - 3.$

Hence, $|e_f(0) - e_f(1)| = 0 < 1$ → (2)

By (1) and (2), $P_4 \odot P_n$ is product cordial.

Case ii) n is even

Correspondingly, all the edges incident with u_i, w_i for i = 2,4,…,n get the label 1.

From figure 4.3, it is clear that two edges are incident with u_i for i = 2,4,…,n-2.

Four edges incident with w_i for i = 2,4,…,n-2 and two edges incident with w_n and one edge is incident with u_n. Further, these edges are all distinct.

Therefore,

$e_f(1) = (2+2+\ldots\frac{n-2}{2}\text{times}) + (4+4+\ldots\frac{n-2}{2}\text{times}) + 2 + 1$

$= 2(\frac{n-2}{2}) + 4(\frac{n-2}{2}) + 3 = (n-2) + 2(n-2) = 3n - 3.$

Further, $e_f(0) = q - e_f(1) = (6n-6) - (3n-3) = 3n-3.$

Hence, $|e_f(0) - e_f(1)| = 0 < 1$ → (3)

By (1) and (3), $P_4 \odot P_n$ is product cordial.

4.4 Theorem: $C_3 \odot C_n$ is not product cordial for $n \geq 3$.

Proof: Let $V(C_3) = \{a_1,a_2,a_3\}$ & $V(C_n) = \{b_1,b_2,\ldots,b_n\}$

Then, $V(C_3 \odot C_n) = \{(a_1,b_j),(a_2,b_j),(a_3,b_j)/j=1,\ldots,n\}$

Label the vertices (a_1,b_j),(a_2,b_j) and (a_3,b_j) with u_j,v_j,w_j for j=1,…,n respectively.

Therefore, $V(C_3 \odot C_n) = \{u_i,v_i,w_i/i=1,\ldots,n\}$.

Correspondingly,

$E(C_3 \odot C_n) = \{u_iv_{i+1},u_iw_{i+1}/i=1,\ldots,n-1\} \cup \{u_nv_1,u_nw_1\}$
$\cup \{u_iv_{i-1},u_iw_{i-1}/i=2,\ldots,n\} \cup \{u_1v_n,u_1w_n\}$
$\cup \{v_iw_{i+1},w_iv_{i+1}/i=1,\ldots,n-1\} \cup \{v_1w_n,v_nw_1\}$.

Here, $|V(C_3 \odot C_n)| = 3n$ and $|E(C_3 \odot C_n)| = 6n$.

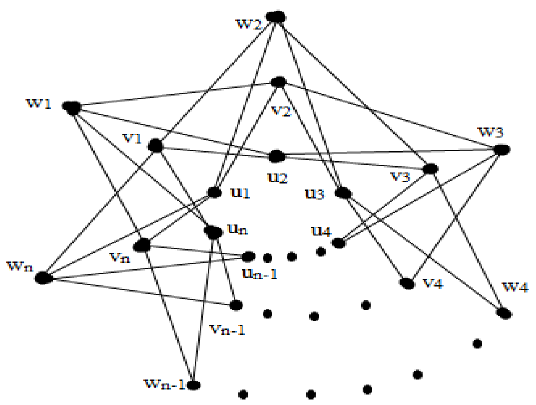


Figure 4.4. $C_3 \odot C_n$

Definef:V(C_3⊙C_n)⟶{0,1}is

by $f(u_i)=\begin{cases}0 & if\ \ i\ \ is\ odd\\ 1 & if\ \ i\ \ is\ even\end{cases}$ and

$f(v_i) = f(w_i) = \begin{cases}0 & if\ \ i\ \ is\ even\\ 1 & if\ \ i\ \ is\ odd\end{cases} \rightarrow \quad (1)$

Case i) n (≥ 3)is odd

Now, label $\lceil 3n/2\rceil$ vertices with 1 and $\lfloor 3n/2\rfloor$vertices with 0, so that $|v_f(0) - v_f(1)| = 1 \longrightarrow \quad (2)$

Equation (1) defines a function f satisfying (1)&gives maximum value for $e_f(1)$.

By the above labeling, $u_2,u_4,\ldots,u_{n-1}$ get the label 1. Corresponding to each of these u_i's4 edges get the label 1 and are distinct.

Also, the vertices $v_1,v_3,v_5,\ldots,v_n$& $w_1,w_3,w_5,\ldots,w_n$ get the label 1. Corresponding to these vertices there are exactly twoedges v_nw_1& v_1w_n get label 1.

All other edges get the label 0.

Therefore,$e_f(1) \leq \sum^{n}_{\substack{i=1\\ i=even}} 4 + 2$ where the summation runs over for even i.

$= \left\lfloor \frac{n}{2} \right\rfloor \times 4 + 2 = 2n.$

$= (\frac{n-1}{2} \times 4) + 2 = 2n - 2 + 2 = 2n.$

Therefore, $e_f(0) \geq (6n) - (2n) = 4n$.

Thus, $|e_f(0) - e_f(1)| \geq |4n - 2n| = 2n > 1$.

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Since f assigns maximum value for $e_f(1)$, there is no other function which is a product cordial labeling of C_3⊙$C_n$when n is odd.

Case ii) n (≥ 4)is even

Now, label 3n/2 vertices with 0 and 3n/2 vertices with 1, so that $|v_f(0) - v_f(1)| = 0 < 1 \longrightarrow \quad (3)$

Equation (1) defines a function f satisfying (1) &gives maximum value for $e_f(1)$.

By the above labeling, $u_2,u_4,\ldots,u_n$ get the label 1. Corresponding to each of these u_i's4 edges get the label 1 and are distinct.

Also, the vertices $v_1,v_3,v_5,\ldots,v_{n-1}$& $w_1,w_3,w_5,\ldots,w_{n-1}$ get the label 1. Corresponding to these vertices there are no edges v_nw_1& v_1w_n get label 1. All other edges get the label 0.

Therefore, $e_f(1) \leq \sum^{n}_{\substack{i=1\\ i=even}} 4$

$= (\frac{n}{2}) \times 4 = 2n.$

Therefore, $e_f(0) \geq (6n) - (2n) = 4n$.

Thus, $|e_f(0) - e_f(1)| \geq |4n - 2n| = 2n > 1$.

Therefore, $|e_f(0) - e_f(1)| \nleq 1$.

Hence, as in case (i), there is no other function which is a product cordial labeling of C_3⊙ C_n when n is even.

By cases (i) &(ii),C_3⊙ C_n is not product cordial.

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